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# EXTENDED ANALYTIC GEOMETRY AS APPLIED TO SIMULTANEOUS EQUATIONS

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**1. Preview.** We begin with a question which goes to the heart of the matter discussed in this paper. Is it true that analytic geometry, with a suitable coordinate system and the aid of algebra, can readily find solutions of some simultaneous equations which are not solvable within reasonable time limits by known purely algebraic methods?

The answer is, or seems to be, "Yes." In some cases, to be sure, the algebra as guided by geometry still remains involved and tedious; but often drastic simplification is possible. To suggest some accomplishments before dealing with details, we shall lead up to one interesting result by considering first the pair

$$(1) \qquad 4x + 3y + 4z + 3u = 25, \qquad \frac{x^2}{7} + \frac{y^2}{11} + \frac{z^2}{18} + \frac{u^2}{14} = 1.$$

Throughout this paper a *solution* of a set of simultaneous equations will be understood to be a common *real* solution, and the set is *consistent* or not according as a solution exists. With this understanding equations (1) are consistent but just barely so, for the *only* solution is (28/25, 33/25, 72/25, 42/25). It will be apparent that an attack based upon the elimination of one letter would lead promptly to an algebraic mess. The example also illustrates the point that it is not always easy to decide whether two equations in more than two unknowns are consistent, quite aside from the matter of producing in that case a real solution.

While the explanation of the solution of (1) must come later, the point of the method need not wait. It will be shown that in one of the many possible coordinate systems the loci of (1) are respectively a line and a "filled circle" to which the line is tangent. From the point of tangency we get at once the sole solution above. And we could have used $n$ unknowns instead of four as in (1), adding for good measure in the new equations $n-2$ parameters in terms of which the lone answer would appear. The point is that one who attacks such problems with algebraic devices alone is handicapping himself unnecessarily, since in effect he is working in the dark, however cleverly.

For maximum usefulness, of course, the geometric methods should be applicable to more general situations in which, for example, a line or curve is not accidentally tangent to another locus. The following theorem shows one of such applications. For simplicity only four unknowns are used, but the generalization to $n$ variables is both obvious and valid. The proof will be deferred to Article 11.

THEOREM 1. *The simultaneous equations*

$$(2) \qquad Ax + By + Cz + Du = E, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{u^2}{d^2} = 1,$$

*are consistent or not according as the line* $X+Y=E$ *does or does not touch the ellipse* $X^2/P^2+Y^2/Q^2=1$, *where* $P^2=a^2A^2+b^2B^2$ *and* $Q^2=c^2C^2+d^2D^2$. *If the line touches the ellipse at* $(X, Y)$ *the corresponding solution is* $(a^2AX/P^2, b^2BX/P^2, c^2CY/Q^2, d^2DY/Q^2)$, *and this will be the only solution if* $(X, Y)$ *is a point of tangency.*

An interesting result is that the first sentence of the theorem is still true if $P^2=a^2A^2$ and $Q^2=b^2B^2+c^2C^2+d^2D^2$, or if the four terms are distributed in any manner between $P^2$ and $Q^2$. The coordinates in the second sentence must be adjusted in an obvious manner in each case. If $P^2=a^2A^2+b^2B^2+c^2C^2+d^2D^2$ the ellipse is replaced by the vertical lines $X^2/P^2=1$, and we get the simple algebraic result:

THEOREM 2. *If* $E=\pm\sqrt{a^2A^2+b^2B^2+c^2C^2+d^2D^2}$, *equations* (2) *of Theorem* 1 *have the sole solution* $(a^2A/E, b^2B/E, c^2C/E, d^2D/E)$.

This gives immediately the previously stated result for equations (1)' though without the possible $n-2$ parameters mentioned below the equations. The obvious generalization to $n$ variables is valid, as for Theorem 1. If we delete the word "sole" Theorem 2 may be verified by direct substitution, and may also be reached by an $n$-space approach. Theorem 1, though slower to apply, gives in addition two specific solutions when the line crosses the ellipse.

To illustrate another type of result, it will be shown that the loci of the two equations

$$(3) \qquad \begin{aligned} &\frac{(x-1)^2}{1} + \frac{(y-2)^2}{2} + \frac{(z+1)^2}{4} + \frac{(u-3)^2}{5} = 1, \\ &\frac{(x+1)^2}{3} + \frac{(y+1)^2}{2} + \frac{(z-1)^2}{5} + \frac{(u-1)^2}{6} = 1 \end{aligned}$$

do not overlap when the axes are chosen properly, and hence the equations are inconsistent. We can reach this conclusion by inspection of (3) by use of an algebraic test for intersection of the "conics" which may be derived from the geometric method.

So much for the general direction and purpose of this article. Since in the development of the basic idea improvements and simplifications have naturally been made, it has seemed advisable to repeat, with appropriate changes, enough introductory material to make this paper self-contained. Additional aspects of the field are dealt with in earlier papers listed at the end of this one.

**2. The three essential features.** The coordinate system which is here called

"extended analytic geometry" provides a generalization of Descartes' system, both plane and solid, which widens the field to $n$ variables instead of just two or three. Its basic features are (1) a set of $n$ axes on a plane which radiate from a common origin and are designated by small letters, (2) a superimposed pair of orthogonal axes denoted by the capitals $X$ and $Y$, and (3) a third-dimensional $Z$-axis through the origin and perpendicular to the $n$-axes plane.

When the first two axes coincide with the $X$ and $Y$ axes, as is often desirable, the first member of the plane sequence, starting with $n=2$, is that of plane analytic geometry, while the corresponding sequence of solid systems begins with solid analytic geometry.

**3. Features of special systems.** While the $X$ and $Y$ axes are inviolate, the small-letter axes may be shifted at will, or even on occasion practically dispensed with. Since it turns out that adaptation of the arrangement of axes to the particular problem is a simple and useful device, it is advisable to discuss more than one system. On the other hand, for easier reading it will be understood that, except where otherwise indicated, *System* A *is used throughout this paper.*

*System* A. Odd-numbered axes coincide with the $X$-axis, and even-numbered ones with the $Y$-axis. The coincidence is in direction as well as position, so that positive directions are to the right or upward. (Fig. 1.)

When not more than six axes are involved, they will be designated in order as $x$, $y$, $z$, $u$, $v$, and $w$, so that $x$, $z$, $v$ are horizontal coordinates. For the case of $n$ axes the letters in order are $x_1, y_1, x_2, \cdots$, where $x_i$ (or $y_i$) stands for the $i$-th horizontal (or vertical) axis, together with its corresponding coordinate. It follows that the fundamental equations of the $n$-axes plane are

$$X = \sum_1^m x_i; \qquad Y = \sum_1^k y_i; \qquad m + k = n; \qquad k = m \text{ or } k = m - 1. \tag{4}$$

The last condition ($k=m$ or $m-1$) is not essential to the theory and is merely incidental to the scheme of numbering axes in System A. It is worth noting that $n-1$ horizontal axes and one vertical one, for example, may be used to advantage in a particular problem, with only obvious and minor changes in the theorems that follow.

*System* B. Here the successive positive directions of the axes are "right, up, left, down, right, up, . . . " and so on. (Fig. 5.) The basic equations are

$$X = x_1 - x_2 + \cdots \pm x_m; \qquad Y = y_1 - y_2 + \cdots \pm y_k. \tag{5}$$

Some consequences of special orientations are noted herewith.

(a) (Applies to A, B, and other variations.) The lattice points of plane (and solid) analytic geometry are preserved. This is useful in dealing with Diophantine equations, for example.

(b) (A and others, but not B.) When positive directions are to the right or upward, only points in the first quadrant can have all-positive coordinates.

(c) (All two-dimensional systems. Special systems of one dimension can be

useful, as will be shown.) The $n$ coordinates of a given point have $n-2$ parameters, or, from another point of view, $n-2$ degrees of freedom.

**4. Generalized coordinates.** From (4) or (5) or the corresponding equations of other systems, we may easily find the *generalized coordinates* (called G.C.) of a fixed point $(X, Y)$. For example, assuming a 5-axes plane, (4) becomes

$$X = x + z + v; \qquad Y = y + u. \tag{6}$$

Therefore the G.C. of $(X, Y)$ are

$$(x, y, z, Y - y, X - x - z), \tag{7}$$

where $x$, $y$, and $z$ are arbitrary. This illustrates (c) above when $n=5$. Note that any three, or, in the general case, any $n-2$, of the coordinates could have been the arbitrary ones. Indeed, we shall have use for the special coordinates

$$(X, Y, 0, 0, \cdots) \tag{8}$$

of the point $(X, Y)$, in which the last $n-2$ coordinates are given the value zero. But it will be seen that 2-axes coordinates, such as (2, 3), for example, are preferable to the G.C., or even to $(2, 3, 0, 0, \cdots)$ for describing a particular point on the $n$-axes plane.

**5. Two types of loci.** The *locus* of an equation, or the totality of points having coordinates which satisfy the equation, is either *normal* (usually an area) or *degenerate* (usually a curve). But a simplification of previous results [3, Articles 5–6] is effected by use of the *revised* definition below.

(9) DEFINITION: *The locus of*

$$f(x_1, y_1, x_2, \cdots) = 0 \tag{10}$$

*is degenerate if and only if* (10) *can be written in the form*

$$F(X, Y) = 0. \tag{11}$$

For example, in the 3-axes case the locus of

$$x^2 + y^2 + z^2 + 2xz = (x + z)^2 + y^2 = X^2 + Y^2 = 0 \tag{12}$$

is degenerate, while the locus of

$$x^2 + y^2 + z^2 = 0 \tag{13}$$

is normal, though each locus consists of a single point (0, 0). Again, if zero is replaced by $-1$ in each of (12) and (13), the loci are imaginary but still degenerate and normal respectively.

**6. Properties of normal and degenerate loci.** Since it is important to distinguish between the separate roles of the two types of loci in dealing with simultaneous equations, we shall state the appropriate theorems.

THEOREM 3. *The locus of the equation* $f(x_1, y_1, x_2, \cdots) = 0$ *is degenerate if and only if*

$$\frac{\partial f}{\partial x_i} \equiv \frac{\partial f}{\partial x_j} \quad \text{and} \quad \frac{\partial f}{\partial y_i} \equiv \frac{\partial f}{\partial y_j} \tag{14}$$

*for every pair* $x_i, x_j$ *and* $y_i, y_j$ *involved.*

For if the locus is degenerate it follows from definition (9) that

$$f = F(X, Y) = F(x_1 + x_2 + \cdots + x_m, y_1 + y_2 + \cdots + y_k). \tag{15}$$

Then

$$\frac{\partial f}{\partial x_i} = \frac{\partial F}{\partial X}\frac{\partial X}{\partial x_i} = \frac{\partial F}{\partial X},$$

which is independent of $i$. Similarly, so is $\partial f/\partial y_i$. Hence (14) holds. Again, since (15) is a solution of (14) $f$ must be expressible in the form (15) when (14) holds, so that the locus is degenerate by the definition.

By way of an example, the locus of

$$f(x, y, z) \equiv 2x^2 + 3xy - y^2 + 2z^2 + 4xz + 3yz - 1 = 0 \tag{16}$$

is degenerate, since $\partial f/\partial x = 4x + 3y + 4z = \partial f/\partial z$.

Note that symmetry of the equation with respect to each pair of like coordinates (both horizontal or both vertical) is a necessary but not sufficient condition for degeneracy of the locus. This is illustrated in equations (12) and (13).

THEOREM 4. *All sets of coordinates of a point on a degenerate locus satisfy the corresponding equation.*

For by (9) the equation may be written in the form $F(X, Y) = 0$, and hence the G.C. of $(X, Y)$ must satisfy it.

THEOREM 5. *The equation of a degenerate locus is obtained in terms of* $X$ *and* $Y$ *merely by replacing the first two variables by* $X$ *and* $Y$ *and the remaining ones by zero.*

For since all sets of coordinates of a point on the locus satisfy the equation, the special set $(X, Y, 0, 0, \cdots)$ must do so.

To illustrate, we go back to (16), whose locus was shown by test to be degenerate. The substitution of Theorem 5 yields

$$f(x, y, z) = F(X, Y) = 2X^2 + 3XY - Y^2 - 1 = 0. \tag{17}$$

Theorems 3 and 5 provide the simple technique for "spotting" equations in $n$ variables which have degenerate loci in System A, and then changing them to equations in two variables with their respective loci unchanged.

THEOREM 6. *At least one but not all of the sets of coordinates of a point on a normal locus satisfy the corresponding equation.*

*Proof.* Without the phrase "but not all," the theorem merely restates the definition of a locus. But if all of the coordinates of $(X, Y)$ satisfy the equation, then the special set $(X, Y, 0, 0, \cdots)$ must do so, and the locus is degenerate rather than normal.

In illustration of Theorem 6, the locus of

$$2x + y + z = 1 \tag{18}$$

is the whole 3-axes plane. But selecting the point (2, 1) at random, we find, upon using the G.C. $(x, 1, 2-x)$, that $(-2, 1, 4)$ are the only coordinates of the point which satisfy (18).

It may be well to point out here that the curves of plane analytic geometry $(n=2)$ are all degenerate according to definition (9), so that Theorem 6 is inapplicable. Theorems 3, 4, and 5 do apply, though trivially.

THEOREM 7. *For each point at which a curve locus touches or crosses an area locus, there is at least one common real solution of the corresponding equations.*

This is a direct corollary of Theorems 4 and 6.

It follows that only the special coordinates of the points described which satisfy the equation of the normal locus need be sought, since these coordinates will surely satisfy the other equation.

THEOREM 8. *If two normal loci have points in common, the corresponding equations may still be inconsistent.*

For example, no two distinct members of the family of equations

$$x + y - z = a \tag{19}$$

have solutions in common, though their common locus is the whole 3-axes plane. But note that if we change to System B, in which $X=x-z$ and $Y=y$, the locus of (19) becomes a family of parallel lines, illustrating the next theorem.

THEOREM 9. *If the loci of two equations have no point in common, the equations have no common real solution.*

For otherwise this solution will locate one and only one point, contradicting the hypothesis.

Obviously the theorem is valid for *any* system of axes which locates one point per set of coordinates (the one system, of course, being used for both equations). We may even, on occasion, [cf. (2)] discard the $n$ axes and let $X$ and $Y$ be single-valued functions of the $n$ coordinates. This wide range of validity of the theorem gives it scope in testing equations suspected of inconsistency. The decision is not always easy—witness (1). But often it is not hard to show that two given equations are consistent. Since in this case the common locus

points cannot be removed by a change of axes, it is often best to proceed at once to the examination of overlapping areas by the methods to be discussed.

**7. A general method for normal loci.** Since normal loci are usually areas on the $n$-axes plane, the technique for picturing them must disclose the bounding curves. One generally applicable method is described here, but often there are shortcuts or canonical forms (Article 8) which make it unnecessary.

If $U$ is a function of $n$ variables, measured in the $Z$-axis direction (*i.e.*, perpendicular to the $n$-axes plane), then *over the point* $(X, Y)$ $U$ is a function of $n-2$ variables plus the constants $X$ and $Y$. The finite absolute maximum or minimum of this latter function, when it exists, may often be found by the method of calculus. It will be the $Z$-value of the surface $Z = F(X, Y)$ which forms the top or bottom of the normally solid $U$-function locus. Paper [3] dealt primarily with such solids and surfaces, but here we are concerned with loci on the $n$-axes plane. So, to get the equation of the bounding curves, we merely set $Z=0$.

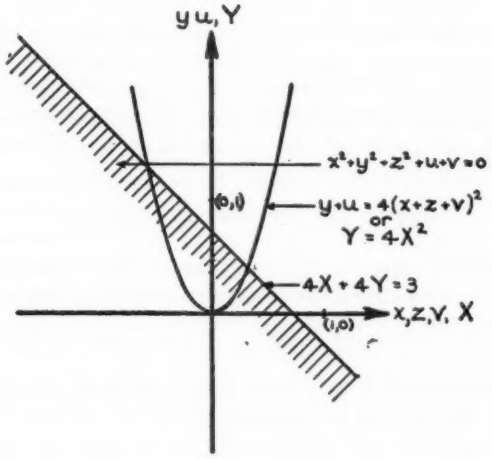


FIG. 1

To illustrate, we shall find the locus on the 5-axes plane of the equation

$$f(x, y, z, u, v) = x^2 + y^2 + z^2 + u + v = 0. \tag{20}$$

By (7) we have

$$\begin{aligned} U &= f(x, y, z, Y - y, X - x - z) \\ &= x^2 + y^2 + z^2 + (Y - y) + (X - x - z). \end{aligned} \tag{21}$$

The necessary conditions for an extremal of $U$, namely $U_x = U_y = U_z = 0$, yield $x=y=z=\frac{1}{2}$, at which $U$ takes the special value represented by $Z$ in the equation,

$$Z = X + Y - 3/4. \tag{22}$$

The $Z$ of the plane (22) is seen to be the *minimum* value of $U$ over the point $(X, Y)$ either by inspection of (21) or by noting that $U_{xx}>0$, $U_{yy}>0$, and $U_{xx}>0$.

In the final step, with $Z=0$, we get the line

$$4X + 4Y = 3 \tag{23}$$

plus all the points on one side of it, as the locus of (20). Since the lower-bound plane (22) lies below the origin, the locus of (20) is the half-plane shown in Figure 1.

Some further play with the locus of Figure 1 may be enlightening. Upon substituting $X=Y=0$ in (21), completing the squares, and equating the result to zero, we get

$$(x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 + (z - \tfrac{1}{2})^2 = \tfrac{3}{4}, \tag{24}$$

so that obviously the origin has sets of real coordinates which satisfy (20). But using $X=1$, $Y=0$, for instance, to get a point on the right of the bounding line, we find that the right side of (24) becomes negative, showing that (1, 0) is not on the locus. Finally, with $Y=(3-4X)/4$ in (21), the right side of (24) becomes zero, and the G.C. of a point $(X, Y)$ on the bounding line are the unique coordinates $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, (1-4X)/4, X-1)$. Hence if we find the equation of a parabola tangent to this line on the right, say

$$16(X - Y)^2 - 88X - 8Y + 61 = 0, \tag{25}$$

for which the point of tangency is $(1, -\frac{1}{4})$, we can say without further calculation that the *one and only* common real solution of (20) and

$$16(x - y + z - u + v)^2 - 88(x + z + v) - 8(y + u) + 61 = 0 \tag{26}$$

is the set of coordinates $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{3}{4}, 0)$. (Try it without the geometry.)

Again, for an application in number theory, let us find common *integral* solutions, if they exist, of (20) and

$$y + u = 4(x + z + v)^2. \tag{27}$$

The locus of (27) is the parabola $Y=4X^2$. Since this crosses the half-plane, (20) and (27) have infinitely many real solutions by Theorem 7. But common integral solutions must be coordinates of the common lattice point (0, 0); in other words, they must be integers $(x, y, z, -y, -x-z)$ such that the values of $x$, $y$, and $z$ satisfy (24). The general method would be to use the lattice points in the locus of (24) on the 3-axes plane, but here it is unnecessary. We simply note that (24) is satisfied in integers when $x$, $y$, and $z$ are assigned the values 0 and 1 independently. Thus there are exactly eight integral solutions of (20) and (27), ranging from the obvious (0, 0, 0, 0, 0) to (1, 1, 1, −1, −2).

**8. Special methods and canonical forms.** When the G.C. of $(X, Y)$ are substituted in an equation we can sometimes see directly that real coordinates of any given point $(X, Y)$ may be found, in which case the locus is the whole plane. For example, given

$$\begin{aligned} 0 &= x^2 + y^2 + z^2 - u^2 - v^2 \\ &= x^2 + y^2 + z^2 - (Y - y)^2 - (X - x - z)^2 \\ &= 2xX + 2yY + 2zX - 2xz - X^2 - Y^2, \end{aligned}$$

we may let $x=z=0$ when $Y\neq 0$, solving for $y$, or let $y=z=0$ when $X\neq 0$, solving for $x$, or let $x=y=z=0$ when $X=Y=0$.

Again, to illustrate another device, consider

$$x^2 + y^2 + z^2 + xy + xz + yz - 1 = 0. \tag{28}$$

Using the G.C. $(x, Y, X-x)$ of $(X, Y)$ in (28) we find that its locus is the ellipse

$$3X^2 + 4XY + 4Y^2 = 4 \tag{29}$$

plus the points inside, since the discriminant of the quadratic in $x$ is negative for points outside.

Probably the most far-reaching of the "inspection" methods for finding loci is the use of canonical forms applying to the general case of $n$ variables. These are illustrated in Article 9.

**9. Generalizations of the conic sections.** The following theorem deals with the plane aspect (with System A instead of B) of Theorem 6 in [3].* It is a generalization to the case of $n$ axes of an ellipse whose center is at the origin. Since the 2-axes formulas for translation of axes carry over easily to the general case, the theorem will enable us to draw readily the loci of all quadratic equations in which each of the $n$ variables appears as a square with a positive coefficient, and no cross-product terms are present, as soon as we have completed the squares and put the equations in the familiar standard form pattern. (An illustration appears near the end of Article 10.)

THEOREM 10. *The locus of the equation*

$$\frac{x_1^2}{a_1} + \frac{y_1^2}{b_1} + \frac{x_2^2}{a_2} + \cdots = 1, \tag{30}$$

*where all denominators are positive and the left side of* (30) *terminates with the n-th term* $(n \geqq 2)$, *is the ellipse*

$$\frac{X^2}{A} + \frac{Y^2}{B} = 1, \qquad A = \sum_1^m a_i, \qquad B = \sum_1^k b_i, \qquad m + k = n, \tag{31}$$

* In the proof of the theorem in [3], $a_i$ should have appeared in the numerator of the expression for $x_i$.

*plus the points inside when $n>2$. Furthermore, the only coordinates of the point $(X, Y)$ on the bounding ellipse* (31) *which satisfy* (30) *are $x_i=a_iX/A$ and $y_i=b_iY/B$.*

*Proof.* Starting with the equation

$$U = -1 + \sum_{i}^{m} \frac{x_i^2}{a_i} + \sum_{1}^{k} \frac{y_i^2}{b_i}, \tag{32}$$

we get, when the values of $x_m$ and $y_k$ from (4) are used,

$$\begin{aligned} U = -1 + \sum_{i}^{m-1} \frac{x_i^2}{a_i} + \sum_{1}^{k-1} \frac{y_i^2}{b_i} + \frac{1}{a_m}(X - x_1 - \cdots - x_{m-1})^2 \\ + \frac{1}{b_k}(Y - y_1 - \cdots - y_{k-1})^2. \end{aligned} \tag{33}$$

Hence

$$\frac{1}{2} U_{x_i} = \frac{x_i}{a_i} - \frac{1}{a_m}(X - x_1 - \cdots - x_{m-1}) \qquad i = 1, 2, \cdots, m-1, \tag{34}$$

with a similar expression for $\frac{1}{2}U_{y_i}$. For a minimum $U$ (obviously it has no maximum) it is necessary that the $m-1$ equations obtained by setting $U_{x_i}=0$ be solved simultaneously. This is accomplished when $x_i=a_iX/A$. Similarly, $y_i=b_iY/B$ for a minimum $U$. Substituting in (32) and replacing $U$ by $Z$ we get

$$\begin{aligned} Z &= -1 + \frac{X^2}{A^2}(a_1 + a_2 + \cdots + a_m) + \frac{Y^2}{B^2}(b_1 + b_2 + \cdots + b_k) \\ &= -1 + \frac{X^2}{A} + \frac{Y^2}{B}. \end{aligned} \tag{35}$$

Thus when $Z=0$, (31) is obtained. Inspection of (33) shows that $U$ has a minimum value over each point $(X, Y)$, and this must be the $Z$ of the surface (35) obtained from the necessary conditions for an extremal.

It should be noted that nothing in the proof requires that the number of horizontal and vertical axes differ by zero or one, as they do in System A. For example, consider the locus of

$$\frac{x^2}{1} + \frac{y^2}{2} + \frac{z^2}{3} + \frac{u^2}{4} = 1. \tag{36}$$

In System A it is the filled ellipse $X^2/4+Y^2/6=1$; if $X=x$, $Y=y+z+u$, the corresponding ellipse is $X^2/1+Y^2/9=1$, where 9 is obtained as the sum of 2, 3, and 4; if $X=x+y+z$, $Y=u$, the ellipse is $X^2/6+Y^2/4=1$; and if $X=x+y+z+u$, $Y=0$, the locus is the line segment on the $X$-axis with endpoints

$X^2/10=1$, or $X=\pm\sqrt{10}$. Again, the locus of

$$\frac{(x-1)^2}{4}+\frac{(y-2)^2}{5}+\frac{(z-3)^2}{6}+\frac{(u+4)^2}{7}=1, \tag{37}$$

for example, is a filled ellipse with center at (1, 2, 3, −4), wherever that point may be. In System A it is the point (4, −2), so that the locus of (37) is then the filled ellipse

$$\frac{(X-4)^2}{10}+\frac{(Y+2)^2}{12}=1. \tag{38}$$

A similar argument, which will be omitted, leads to

THEOREM 11. *The locus of the equation*

$$\sum_1^k y_i=\sum_1^m \frac{x_i^2}{a_i}, \qquad a_i>0, \tag{39}$$

*is the parabola*

$$Y=\frac{X^2}{A}, \qquad A=\sum_1^m a_i \tag{40}$$

*plus the points inside when* $m>1$. *The coordinates of the point* $(X, Y)$ *on* (40) *which satisfy* (39) *are:* $x_i=a_iX/A$, *with* $y_i$ *subject only to the condition that* $\sum_1^k y_i=Y$.

The generalizations of the hyperbola are not so simple. When we allow some of the denominators in (30) to be negative and the curve (31) exists, it is sometimes but not always a boundary of the locus. In any case it is a useful test curve, and the locus is found easily by testing points on opposite sides of the curve to see whether the locus extends across it. For example, the locus of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-\frac{u^2}{d^2}=1, \qquad a^2>c^2,\ b^2>d^2, \tag{41}$$

is an ellipse plus the points *outside*, the ellipse being

$$\frac{X^2}{a^2-c^2}+\frac{Y^2}{b^2-d^2}=1. \tag{42}$$

To illustrate other possibilities, the locus of

$$2x+3y+4z=\frac{(u-1)^2}{4}+\frac{(v-2)^2}{5} \tag{43}$$

is "by inspection" the filled parabola

$$Y = \frac{(X-3)^2}{9}, \quad \text{if} \tag{44}$$

$$X = u + v; \qquad Y = 2x + 3y + 4z. \tag{45}$$

However, a somewhat complicated transformation like (45) should in general be used for a set of simultaneous equations only when it makes one locus degenerate—always a desirable goal.

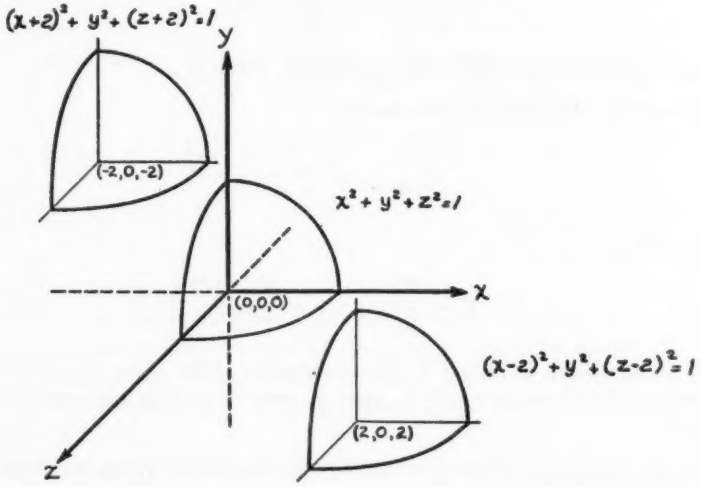


FIG. 2. Three dimensions.

We are now ready to consider equations (1) again, using System A. By Theorems 3 and 5 the first locus is that of the line $4X+3Y=25$, while by Theorem 10 the second locus is the filled circle $X^2/(7+18)+Y^2/(11+14)=1$, or $X^2+Y^2=25$. The line is tangent to the circle at (4, 3), and again by Theorem 10 the only coordinates satisfying both equations are (28/25, 33/25, 72/25, 42/25). This, as we have seen, could also be obtained from Theorem 2. But now more appears. For if the denominators are replaced by $a$, $b$, $25-a$ and $25-b$ respectively (all positive), the filled circle locus remains unchanged, and the unique solution in terms of the parameters becomes $(4a/25,\ 3b/25,\ (100-4a)/25,\ (75-3b)/25)$. Evidently $n$ unknowns instead of four could have been used. For another variation of this theme we can say that the equations

$$5x - 9y + 5z - 9u = -7, \qquad \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{25-a} + \frac{u^2}{25-b} = 1, \tag{46}$$

with $a$ and $b$ arbitrary except that the denominators are positive, have infinitely many real solutions since the line crosses the filled circle, but no integral solutions since the chance for this is lost at the one common lattice point (4, 3).

**10. "Parallax" in loci.** Consideration of the loci of

$$x^2 + y^2 + z^2 = 1, \tag{47}$$

$$(x - 2)^2 + y^2 + (z - 2)^2 = 1, \quad \text{and} \tag{48}$$

$$(x + 2)^2 + y^2 + (z + 2)^2 = 1 \tag{49}$$

in solid analytic geometry will suggest why a change of axes sometimes serves a useful purpose. In Figure 2 the spherical surfaces of the three-dimensional

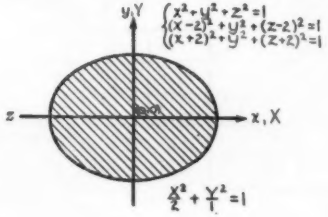


FIG. 3. System B.

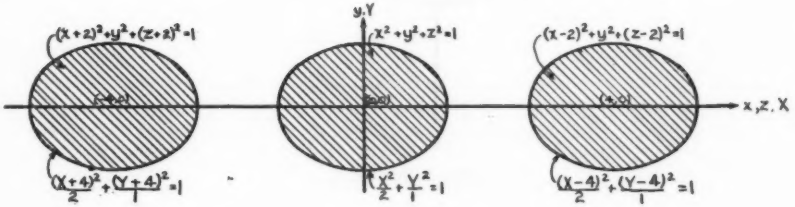


FIG. 4. System A.

system are shown. Now suppose the observer to be located "at infinity" in the direction from $O$ which bisects the plus-$x$ and plus-$z$ axes. The axes would then appear as in Figure 3, and we would have System B, with $X = x - z$. The three centers would merge at the origin and the loci would be coincident filled ellipses, though the equations are inconsistent since the spheres of Figure 2 do not intersect. But if the point of view for Figure 2 were at infinity on the line bisecting the plus-$z$ and minus-$x$ axes, the loci should be separated. This gives us Figure 4, with System A, and the prediction is verified, though the projection boundaries in Figures 3 and 4 are not circles as we might have expected.

Thus the loci of equations on the 3-axes plane may be considered as projections or silhouettes of the surfaces of solid analytic geometry, and knowledge of these surfaces helps us to predict the forms of the loci. The projections of

quadric surfaces will include, for example, curves (from cylindrical surfaces "seen on end"), "windows," (looking endwise through a hyperboloid of one sheet) and in fact all types of closed or open areas which are bounded, if at all, by conic sections. And this, incidentally, completes the list of possible types of quadratic loci on the $n$-axes plane.

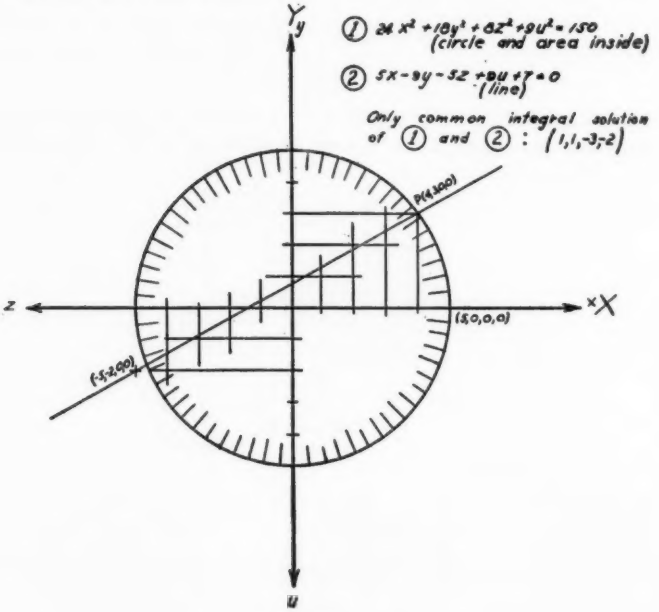


FIG. 5. System B.

In the case of four or more variables to say that "when we change the axes we change the point of view" is of course only a figure of speech, but it is helpful. To illustrate, let us return to equations (3). In System A the two loci, by the discussion preceding (37), are the filled ellipses $X^2/5+(Y-5)^2/7=1$ and $X^2/8+Y^2/8=1$, which certainly overlap. But when we observe the second terms, namely $(y-2)^2/2$ and $(y+1)^2/2$, we note that if

$$X = x + z + u; \qquad Y = y, \tag{50}$$

the vertical distance 3 between the centers is more than the sum of the semi-minor axes, and the loci cannot overlap. The equations of the boundaries (Fig. 7) are then

$$\frac{(X-3)^2}{10}+\frac{(Y-2)^2}{2}=1, \qquad \frac{(X-1)^2}{14}+\frac{(Y+1)^2}{2}=1. \tag{51}$$

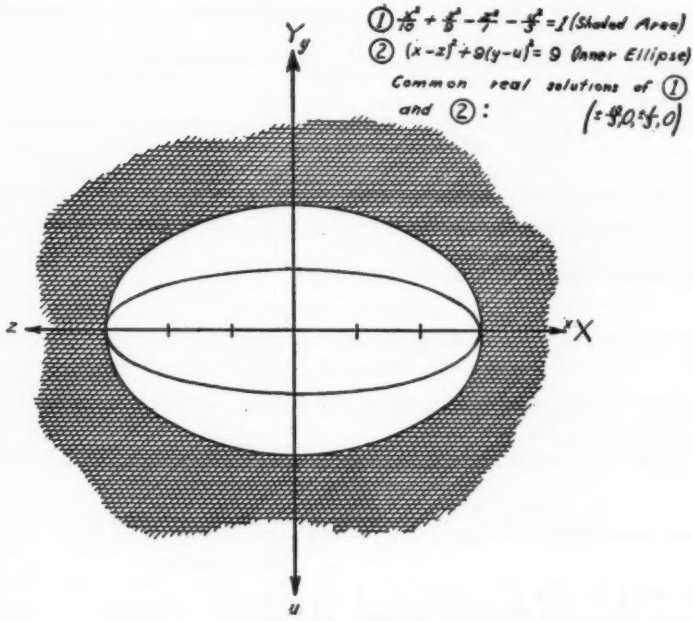


FIG. 6. System B.

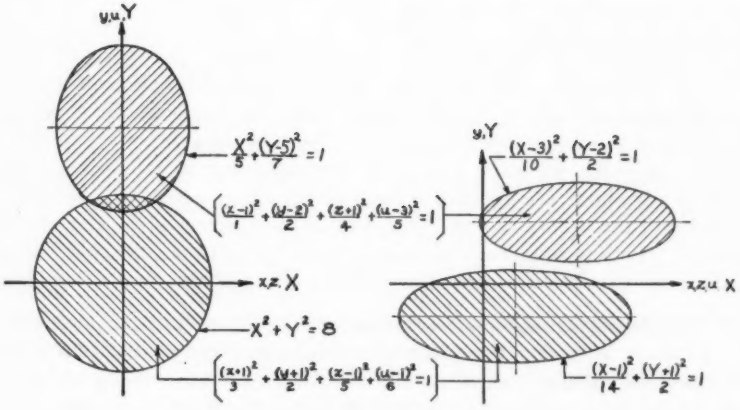


FIG. 7

This illustration points to a generalization, thus:

THEOREM 12. *Necessary conditions for the consistency of*

$$\sum \frac{(x_i - a_i)^2}{c_i^2} = 1, \qquad \sum \frac{(x_i - b_i)^2}{d_i^2} = 1, \qquad c_i > 0,\ d_i > 0 \tag{52}$$

*are:* (a) $|a_i - b_i| \leqq c_i + d_i$ *for each* $i$, *and* (b) $|\sum a_i - \sum b_i| \leqq \sqrt{\sum c_i^2} + \sqrt{\sum d_i^2}$ *where all summations are from* $i = 1$ *to* $i = n$.

The second necessary condition appears if we let

$$X = \sum_1^n x_i; \qquad Y = 0, \tag{53}$$

so that the loci are line segments on the $X$-axis with centers respectively at $(\sum_1^n a_i,\ 0)$ and $(\sum_1^n b_i,\ 0)$.

An application appears even in plane analytic geometry. The ellipses

$$\frac{(x+1)^2}{9} + \frac{y^2}{4} = 1, \qquad \frac{(x-3)^2}{5} + \frac{(y-3)^2}{5} = 1 \tag{54}$$

do not touch by test (b), since $|(-1+0)-(3+3)| > \sqrt{13}+\sqrt{10}$, though the decision is close and tediously attained by the usual methods.

**11. A complete solution.** We now return to equations (2). The reader will note that the axes-systems thus far discussed will translate an equation of the first degree into a plane usually, and into a line only exceptionally and for special coefficients. The locus of the second equation in (2) is stubbornly a filled ellipse in almost any set-up. But if we let $X = Ax + Cz$ and $Y = By + Du$ (or, in the general case, $X = \sum_1^m A_i x_i$, $Y = \sum_1^k B_i y_i$) we get the line $X + Y = E$, and our general method for finding the other locus still applies. There is still one point $(X, Y)$ to correspond with any given set of coordinates. The routine method gives the result for (2) as already stated, and the details of the proof for the generalization to $n$ variables are similar to those in the proof of Theorem 10.

More generally, we have here a method applying to linear-and-quadratic pairs in $n$ variables. We make the locus of the linear equation a straight line, and then find the other locus as it is constrained to be under the required transformation. The line touches or does not touch the other locus according as the equations are consistent or not. Various extensions of the method are possible.

**12. Concluding remarks and summary.** It is evident that simultaneous solutions of two or more equations in $n$ unknowns are greatly facilitated if at least one locus is degenerate, so that we may deal with curves crossing areas or each other. Whenever we can gain this end by adjusting the transformation to one of the equations, we can afford to discard canonical forms and let the other

loci be what they may be as found by the general method. For example, consider

$$U \equiv 4x^2 + y^2 + z^2 + u^2 - 4xy + 4xz - 2yz - 1 = 0. \tag{55}$$

Having in mind Theorem 3, with similar extensions to axes-systems other than $A$, we note first that

$$\begin{aligned} U_x &= 8x - 4y + 4z = \quad 4(2x - y + z); \\ U_y &= 2y - 4x - 2z = -2(2x - y + z); \\ U_z &= 2z + 4x - 2y = \quad 2(2x - y + z); \\ U_u &= 2u. \end{aligned}$$

This method suggests, as it may in less obvious cases, a more useful version of the first equation. We see that (55) can be written thus:

$$(2x - y + z)^2 + u^2 = 1. \tag{56}$$

An efficient transformation appears at once, namely:

$$X = 2x - y + z; \qquad Y = u. \tag{57}$$

In the equations of transformation $X$ and $Y$ may advantageously share some small letters. For example, given

$$(3x + 2y - 4z)^2 + (3x + 2y - 4z)(2x - y + z) = 1, \tag{58}$$

the locus is obviously a hyperbola if we let

$$X = 3x + 2y - 4z, \qquad Y = 2x - y + z. \tag{59}$$

The G.C. of $(X, Y)$ are then $(x, (11x - X - 4Y)/2, (7x - X - 2Y)/2)$. No essential change in the general method is necessary.

Let us consider briefly the "worst possible" case, involving $n$ consistent equations in $n$ unknowns, for which the loci in all convenient systems overlap in a common area. Suppose that we then use System A. By eliminating the $n-2$ small letters from $n-1$ of the equations, leaving only $X$ and $Y$, we get a "curve of intersection." A second curve is obtained from $n-1$ equations including the previously neglected one, and the two equations in $X$ and $Y$ are solved simultaneously. When the G.C. of a point of intersection are substituted in the original equations two of the latter will be dependent on the others, and then the process is repeated with fewer equations and unknowns. This of course is merely a variation of the usual (often hopeless) algebraic approach; but the geometric leads may suggest shortcuts at any point.

Again, if the common locus area is finite, all integral solutions may be found by a systematic examination of lattice points.

We have seen (Theorem 1) that if one equation is linear its locus may be made a line. If all equations are linear various simplifications have been suggested in previous papers. If two of the equations are inconsistent, this shows

up quickly in many cases (Theorem 12).

In some situations algebra must still do the main job. For example, when one "$n$-variable ellipsoid" is completely "inside" another, the positions of the plane loci will suggest this, but the simplest proof of inconsistency is likely to be found in the use of inequalities.

To summarize, whereas in the case of simultaneous quadratics in $n$ unknowns the algebra of the solution is always adequate when $n=2$, so that the geometric illustrations found in most algebra texts are incidental and unnecessary, though helpful as checks, the situation is practically reversed when $n>2$. Then geometry takes over the dominant role in many cases, while algebra hangs on merely as the necessary clerical help.

**References**

Earlier Papers by the Author on Extended Analytic Geometry in this MONTHLY

1. An analytic geometry for $N$ variables, vol. 52, 1945, pp. 253–262.
2. Some applications of extended analytic geometry, vol. 56, 1949, pp. 158–164.
3. Functions of $N$ variables in extended analytic geometry, vol. 59, 1952, pp. 453–460.

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## THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the fourteenth William Lowell Putnam Mathematical Competition held on March 6, 1954, have been determined in accordance with the constitution of the Competition. This Competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Cornell University, Ithaca, New York. The members of the team were Leonard Evens, D. J. Kleitman and Steven Weinberg; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were David Lewin, Benjamin Muckenhaupt and Kenneth Wilson; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were George H. Borrmann, Jr., Max A. Plager, and Kenneth E. Ralston; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario. The members of the team were Marcus T. Grisaru, William M. Kahan and Charles B. H. Watson; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are James D. Bjorken, Massachusetts Institute of Technology; Leonard Evens, Cornell University; William P. Hanf, University of California (Berkeley); Benjamin Muckenhaupt, Harvard University; and Kenneth Wilson, Harvard University. Each of these will receive a prize of fifty dollars.

The five succeeding persons ranking highest in the examination, named in alphabetical order, are Benito Franqui, University of Puerto Rico; Angus C. Kerr-Lawson, University of Toronto; J. S. Lew, Yale University; David Mumford, Harvard University; and Jack Towber, Brooklyn College. Each of these will receive a prize of twenty dollars.

The following teams, named in alphabetical order, won honorable mention: California Institute of Technology, Pasadena, California, the members of the team being George A. Baker, Jr., David G. Cantor, and Robert D. Ryan; Columbia College, New York, New York, the members of the team being Lee Abramson, Elihu Lubkin, and Richard Wasserman; Kenyon College, Gambier, Ohio, the members of the team being Trevor Barker, Robert G. Busacker, and David Ryeburn; Swarthmore College, Swarthmore, Pennsylvania, the members of the team being John Hopfield, Paul Monsky, and Lisa Steiner.

Twelve individuals were given honorable mention. The names are listed in alphabetical order: George A. Baker, Jr., California Institute of Technology; Trevor Barker, Kenyon College; Edward Curtis, Harvard University; John E. Derwent, University of Notre Dame; Donald Fredkin, New York University; Sidney Kahana, University of Manitoba; William M. Kahan, University of Toronto; Elihu Lubkin, Columbia College; Paul Monsky, Swarthmore College; Kenneth Ralston, Massachusetts Institute of Technology; Steven Weinberg, Cornell University; and Harold Weitzner, University of California (Berkeley).

The following is a list of all colleges and universities which entered teams in the Competition. The list, in alphabetical order, is: Agricultural and Mechanical College of Texas, Arizona State College (Flagstaff), Arizona State College (Tempe), Brooklyn College, Brown University, California Institute of Technology, Carleton College, Carnegie Institute of Technology, College of the Holy Cross, College of Saint Catherine, Columbia College (New York), Cornell University, Doane College, Florida Agricultural and Mechanical University, Harvard University, Iowa State College, Kent State University, Kenyon College, Knox College, Lebanon Valley College, Massachusetts Institute of Technology, McGill University, McMaster University, Queen's College (Flushing, N. Y.), Queen's University (Kingston, Ontario), Stanford University, Swarthmore College, Syracuse University, Tennessee Agricultural and Industrial State College, Texas College (Tyler), The College of the City of New York, The

Cooper Union, United States Naval Academy, Université de Montréal, University of British Columbia, University of California, University of Georgia, University of Manitoba, University of Minnesota, University of Notre Dame, University of Oregon, University of Puerto Rico, University of Rochester, University of the South, University of Toronto, University of Washington, Ursinus College, Yale University.

The following additional colleges and universities entered individual contestants only: Bethel College, Florida Southern College, Huron College, Nebraska State Teachers College, New York University, North Georgia College, Oberlin College, Purdue University, Regis College, Saint Joseph College, Saint Olaf College, University of California (Berkeley), University of Colorado, University of Kentucky, University of Michigan, University of New Mexico, University of Oklahoma, Washington University, Western Washington College of Education.

A total of 231 undergraduates representing 67 institutions took part in the Competition.

The departments of mathematics of any of the competing institutions may obtain the rankings of their individual contestants (except that the relative rankings of the first five will not be divulged) by writing to the Director of the Competition, Room 301 Merrill Hall, Kent State University, Kent, Ohio. These rankings may now be given to the individual contestants by their own departments of mathematics. Any other departments of mathematics may obtain the individual rankings of contestants for the purpose of selecting graduate students.

Participants in the Competition were given the following lists of problems:

### Part I

MARCH 6, 1954

MORNING SESSION: 9:00 A.M. TO 12:00 NOON

*Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate all answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.*

*Omit one question. You must indicate which question is omitted.*

1. Let $n$ be an odd integer greater than 1. Let $A$ be an $n$ by $n$ symmetric matrix such that each row and each column of $A$ consists of some permutation of the integers $1, \cdots, n$. Show that each one of the integers $1, \cdots, n$ must appear in the main diagonal of $A$.

2. Consider any five points $P_1, P_2, P_3, P_4, P_5$ in the interior of a square $S$ of side-length 1. Denote by $d_{ij}$ the distance between the points $P_i$ and $P_j$. Prove that at least one of the distances $d_{ij}$ is less than $\sqrt{2}/2$. Can $\sqrt{2}/2$ be replaced by a smaller number in this statement?

3. Prove that if the family of integral curves of the differential equation

$$\frac{dy}{dx} + p(x)y = q(x) \qquad p(x)\cdot q(x) \not\equiv 0$$

is cut by the line $x=k$, the tangents at the points of intersection are concurrent.

4. A uniform rod of length $2k$ and weight $w$ rests with the end $A$ against a smooth vertical wall, while to the lower end $B$ is fastened a string $BC$ of length $2b$ coming from a point $C$ in the wall directly above $A$. If the system is in equilibrium, determine the angle $ABC$.

5. If $f(x)$ is a real-valued function defined for $0<x<1$, then the formula $f(x) = o(x)$ is an abbreviation for the statement that

$$\frac{f(x)}{x} \to 0 \qquad \text{as } x \to 0.$$

Keeping this in mind, prove the following: if

$$\lim_{x\to 0} f(x) = 0 \quad \text{and} \quad f(x) - f\left(\frac{x}{2}\right) = o(x),$$

then $f(x) = o(x)$.

6. Suppose that $u_0, u_1, u_2 \cdots$ is a sequence of real numbers such that

$$u_n = \sum_{k=1}^{\infty} u_{n+k}^2 \qquad \text{for } n = 0, 1, 2, \cdots.$$

Prove that if $\sum u_n$ converges then $u_k = 0$ for all $k$.

7. Prove that there are no integers $x$ and $y$ for which

$$x^2 + 3xy - 2y^2 = 122.$$

Part II

AFTERNOON SESSION: 2:00 TO 5:00 P.M.

*Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate all answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.*

*Omit one question. You must indicate which question is omitted.*

1. Show that the equation $x^2 - y^2 = a^3$ has always integral solutions for $x$ and $y$ whenever $a$ is a positive integer.

2. Assume as known the (true) fact that the alternating harmonic series

(1) $\quad 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + 1/7 - 1/8 + \cdots$

is convergent, and denote its sum by $s$. Rearrange the series (1) as follows:

(2) $\quad 1 + 1/3 - 1/2 + 1/5 + 1/7 - 1/4 + 1/9 + 1/11 - 1/6 + \cdots$.

Assume as known the (true) fact that the series (2) is also convergent, and denote its sum by $S$. Denote by $s_k$, $S_k$ the $k$th partial sum of the series (1) and (2) respectively. Prove the following statements.

(a) $\quad S_{3n} = s_{4n} + \frac{1}{2} s_{2n}$.

(b) $\quad S \neq s$.

3. Let $a$ and $b$ denote real numbers such that $a<b$. The symbol $(a, b)$ will denote the closed interval with the end points $a$, $b$. Let there be given a collection of closed intervals $(a_1, b_1), \cdots, (a_n, b_n)$ such that any two of these closed intervals have at least one point in common. Prove that there exists then a point which is contained in every one of these intervals.

4. Given the focus $f$ and the directrix $D$ of a parabola $P$ and a line $L$, describe (with proof) a Euclidean (*i.e.* ruler and compass) construction of the point or points of intersection of $L$ and $P$. Be sure to identify the case for which there are no points of intersection.

5. Let $f(x)$ be a real-valued function, defined for $-1<x<1$, such that $f'(0)$ exists. Let $a_n$, $b_n$ be two sequences such that

$$-1 < a_n < 0 < b_n < 1, \qquad \lim_{n\to\infty} a_n = 0, \qquad \lim_{n\to\infty} b_n = 0.$$

Prove that

$$\lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n} = f'(0).$$

6. Prove that every positive rational number is the sum of a finite number of distinct terms of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

7. Show that

$$\lim_{n\to\infty} \sum_{s=1}^{n} \left(\frac{a+s}{n}\right)^n \qquad (a > 0)$$

lies between $e^a$ and $e^{a+1}$.

### Solutions of the Problems*

The following solutions are not taken from any of the contestants' papers, but generally embody ideas used by many contestants. The presentation here is intended as a brief sketch of the method of proof rather than as a model of a detailed proof such as is expected from the contestants.

## Part I

1. Any integer $j$ such that $1 \leqq j \leqq n$ occurs just once in each row and hence occurs an odd number ($n$) times in the matrix. By the symmetry of the matrix the number of off-diagonal occurrences of $j$ is even and hence $j$ must occur on the diagonal.

2. Divide the open square $S$ into four congruent half-open squares. Two points, say $P_1$, $P_2$, must belong to one of the half-open squares. The components of the vector $\overrightarrow{P_1P_2}$ parallel to the sides of the square $S$ are then each numerically less than $1/2$, and thus the length of the vector $\overrightarrow{P_1P_2}$ is less than $1/\sqrt{2}$.

Since $P_5$ may be taken at the center of $S$ and $P_1$, $P_2$, $P_3$, $P_4$ placed arbitrarily close to the four vertices of $S$ it follows that the theorem is false if a smaller number is substituted for $1/\sqrt{2}$.

3. The equation of the tangent line to the integral curve at the point $(k, c)$ is $y-c=[q(k)-cp(k)](x-k)$ and this equation is satisfied by the point, $(k+1/p(k), q(k)/p(k))$ which does not depend on $c$.

4. Let $F$, $T$, and $R$ be vectors representing the forces acting on the rod due to gravity, the tension of the string, and the reaction of the wall. By the principle of moments these vectors must be concurrent for equilibrium to obtain. If $B$ is not on the wall the point $D$ where these forces concur is the midpoint of $BC$. Hence $4k^2+4b^2-8bk\cos\theta=(AC)^2=b^2-(AD)^2=b^2-(4k^2+b^2-4bk\cos\theta)$ and so $\cos\theta=2k/3b+b/3k$. If $b<k$ or $b>2k$, this is impossible, and then $B$ is on the wall and $\theta=0$.

5. Given $\omega>0$ there is a $\phi>0$ such that $|f(x)-f(x/2)|<\omega|x|$ for $0<|x|<\phi$. Then for $0<|x|<\phi$ we have $|f(x)-f(x/2^n)| \leqq \sum_{j=0}^{n-1} |f(x/2^j)-f(x/2^{j+1})| < \sum_{j=0}^{n-1} \omega|x|/2^{j+1} < \omega|x|$ for every positive integer $n$. Since $n$ is arbitrary we have $|f(x)| \leqq \omega|x|$ for $0<|x|<\phi$.

6. Clearly $0 \leqq u_{n+1} \leqq u_n$, $n=1, 2, \cdots$. If $\sum u_j$ converges we may take $k \geqq 1$ so that $\sum_{k+1}^{\infty} u_j < 1$. Then $u_{k+1} \leqq u_k = \sum_{k+1}^{\infty} u_j^2 \leqq u_{k+1} \sum_{k+1}^{\infty} u_j \leqq u_{k+1}$. Hence $u_k = u_{k+1}$ and so $u_{k+1}=0$, implying $u_j=0$ for $j>k$ since $u_{k+1}=\sum_{k+2}^{\infty} u_j^2$. By finite induction $u_j=0$ for $j<k+1$.

7. If $x$ and $y$ are integers then $17y^2+488=k^2$ for some integer $k$. Hence

* These solutions are published solely for the information of interested persons. Neither the editor, nor the director of the competition, nor the paper grader will enter into any correspondence concerning them.

$k^2 \equiv 488 \equiv 12 \pmod{17}$. By examination of a complete residue class, or by the theory of quadratic residues, this is shown to be impossible.

## Part II

1. Putting $x+y=a^2$ and $x-y=a$ we have the solutions $x=(a^2+a)/2$, $y=(a^2-a)/2$ which are clearly integers when $a$ is an integer.

2. Evidently $S_2=s_4+s_2/2$. Since

$$S_{3n} = \sum_{j=1}^{2n} \frac{1}{2j-1} - \sum_{j=1}^{n} \frac{1}{2j}$$

then, by induction,

$$\begin{aligned} S_{3n+3} &= S_{3n} + \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2} \\ &= s_{4n} + s_{2n}/2 + \frac{1}{4n+1} - \frac{1}{4n+2} + \frac{1}{4n+3} - \frac{1}{4n+4} + \frac{1}{4n+2} - \frac{1}{4n+4} \\ &= s_{4n+4} + s_{2n+2}/2. \end{aligned}$$

Since the series are given convergent $S=3s/2$. Since

$$s = \sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{1}{2n} \right) > 0$$

it follows that $S \neq s$.

3. Since $(a_j, b_j)$ and $(a_k, b_k)$ have a point in common we have $a_j \leqq b_k$, $j, k = 1, 2, \cdots, n$. Thus $m = \max [a_1, \cdots, a_n]$ satisfies $a_j \leqq m \leqq b_j$ for $j = 1, \cdots, n$. The property also holds for an infinite collection of intervals if we take $m = \sup (a_j)$.

4. Let $p$ be any point on $L$ but not on $D$. Construct a circle with center $p$ and tangent to $D$ at $m$. Construct line $F$ through $f$ and the intersection of $L$ and $D$ (if $L$ and $D$ are parallel take $F$ parallel to $D$). Let the above circle intersect $F$ in $r$, $s$. Take points $t$, $u$ on $L$ so that $ft$ is parallel to $ps$ and $fu$ is parallel to $pr$. Then $t$ and $u$ are the required points. If $r$ and $s$ coincide then $t$ and $u$ coincide and the parabola intersects $L$ in just the point $t$. If the circle does not cut $F$ the parabola does not cut $L$. The proof follows by consideration of similar triangles.

5. Let $r+s=1$, $r>0$, $s>0$, and $h \leqq k$ be real numbers. Then $h \leqq rh+sk \leqq k$. That is, a weighted mean of two real numbers lies between the numbers. Hence

$$\frac{f(b_n) - f(a_n)}{b_n - a_n} = \frac{f(b_n) - f(0)}{b_n} \cdot \frac{b_n}{b_n - a_n} + \frac{f(0) - f(a_n)}{0 - a_n} \cdot \frac{-a_n}{b_n - a_n}$$

and so $(f(b_n)-f(a_n))/(b_n-a_n)$ lies between

$$(f(b_n)-f(0))/b_n \text{ and } (f(0)-f(a_n))/(0-a_n).$$

Since the latter two fractions approach $f'(0)$ as $n\to\infty$ so then must the fraction

$$(f(b_n)-f(a_n))/(b_n-a_n).$$

6. Let $A$ and $B$ be positive integers. Then by the divergence of the harmonic series there is a unique non-negative integer $n_0$ such that $\sum_{j=1}^{n_0} 1/j < A/B \leqq \sum_{j=0}^{n_0+1} 1/j$. (The $\sum_{j=1}^{0}$ is taken as 0.) If equality holds the desired representation is at hand and so we assume $A/B < \sum_{j=0}^{n_0+1} 1/j$. Then $A/B - \sum_{j=1}^{n_0} 1/j = C/D < 1/(n_0+1)$. Take $n_1$ as the unique positive integer such that $1/(n_1+1) \leqq C/D < 1/n_1$. We again suppose inequality as the problem is otherwise solved, and put $C/D - 1/(n_1+1) = E/F > 0$. But $E/F = (C(n_1+1)-D)/D(n_1+1)$ and $C(n_1+1)-D<C$ so that with $E/F$ in lowest terms we must have $E<C$. Then $E/F < 1/n_1(n_1+1)$ and so the unique integer $n_2$ such that $1/(n_2+1) \leqq E/F < 1/n_2$ satisfies $n_2 > n_1$. In a finite number of steps we must obtain the desired representation, since if the equality does not occur before, it must occur when the numerator of the reduced fraction has become 1.

7. For $|x|<n$, $(1+x/n)^n \leqq e^x$. Hence for $n>a$ we have

$$\sum_{s=1}^{n}\left(\frac{a+s}{n}\right)^n = \sum_{r=0}^{n-1}\left(1+\frac{a-r}{n}\right)^n < \sum_{r=0}^{\infty} e^{a-r} = \frac{e^{a+1}}{e-1}$$

and so

$$\overline{\lim} \sum_{s=1}^{n}\left(\frac{a+s}{n}\right)^n \leqq \frac{e^{a+1}}{e-1}.$$

For any positive integer $k$ and $n>a+k$

$$\sum_{s=1}^{n}\left(\frac{a+s}{n}\right)^n \geqq \sum_{r=0}^{k}\left(1+\frac{a-r}{n}\right)^n$$

so that

$$\underline{\lim} \sum_{s=1}^{n}\left(\frac{a+s}{n}\right)^n \geqq \sum_{r=0}^{k} e^{a-r}.$$

Since this is true for all positive integers $k$ we must have

$$\underline{\lim} \sum_{s=1}^{n}\left(\frac{a+s}{n}\right)^n \geqq \sum_{r=0}^{\infty} e^{a-r} = \frac{e^{a+1}}{e-1}.$$

Since

$$e^a < \frac{e^{a+1}}{e-1} < e^{a+1}$$

the proof is complete.

# MATHEMATICS FOR SOCIAL SCIENTISTS*

R. R. BUSH, W. G. MADOW, HOWARD RAIFFA, R. M. THRALL†

**1. Introduction. (Madow).** The 1953 Summer Institute of Mathematics for Social Scientists met for eight weeks in the summer of 1953 under the sponsorship of the Social Science Research Council with the help of a grant from the Behavioral Sciences Division of the Ford Foundation.

Approximately twenty hours weekly were devoted to lectures by the four members of the faculty. In addition, ten hours weekly were devoted to supervised study and supplementary lectures.‡ Finally, eight visitors gave eighteen lectures on applications of mathematics in the social sciences that were not discussed in the regular lectures. The Institute was neither a conference nor a discussion group. It was simply a stiff summer session in which topics in mathematics and applications of mathematics in the social sciences were taught intensively.

Although the nature of the student body and the intensity of the program keep the Institute from being considered to be a model for an undergraduate mathematics curriculum, yet the subject matter and level of the curriculum of the Institute were such that we believe the Institute to be a suitable subject for discussion here.

Of late years considerable dissatisfaction has developed with the present fairly standardized first two years undergraduate mathematics curriculum. The reasons for this dissatisfaction are well known and need no repetition. The dissatisfaction is felt not only by social scientists but by others; perhaps most of all, by mathematicians. The Mathematical Association of America has at least one committee working in this area.

We feel that a two year undergraduate curriculum based largely upon the subject matter of the 1953 Summer Institute can be successfully taught not only to majors in social sciences but also, after minor revision, to all students except engineers and physical scientists. Actually, this curriculum might even meet the needs of engineers and physical scientists, but we think it would be practical first to organize this curriculum as an alternative to, rather than replacement of, the present curriculum.

For admittance to the Institute, students were required to be social scientists and to have had at least one semester of college mathematics or its equivalent in independent study. Over 250 persons applied for admission. Thirty persons were admitted and given grants. Twenty-six persons were admitted without

* This is a report of a round table discussion that occurred at the meeting of the Mathematical Association of America at Johns Hopkins University, December 31, 1953.

† The faculty of the 1953 Summer Institute consisted of W. G. Madow, University of Illinois, director, R. R. Bush, Harvard University, Howard Raiffa, Columbia University, and R. M. Thrall, University of Michigan, assisted by R. L. Davis, University of Michigan and R. E. Priest, University of Illinois.

‡ These lectures were given by Davis and Priest.

grants. Of the 56 persons thus admitted 41 attended the Institute.

Twenty-two of the students had received the Ph.D. degree, 12 of them having academic rank of assistant professor or higher. The other 19 were graduate students in a social science.

Twenty-four of the students were between 22 and 30 years of age; the remaining 17 were between 31 and 42 years of age.

The students came from 23 institutions in 13 states. There were 20 psychologists, 11 sociologists, 7 economists, and 3 political scientists.

Twenty-two of the students had studied some calculus. Nineteen had not. But the training in calculus was often weak and had not been used for some years.

The students were highly selected not only by the fact they were admitted to the Institute but also by the fact that they applied. They were very highly motivated and were mature responsible people who had given evidence of achievements in their own fields.

It is still early to state how successful we were. Of the 41, only two did really poorly and only five did fairly poorly. We were well satisfied with the others. But it is only as our students begin using their work of this summer that we will begin to be able to evaluate this summer's work.

No summer institutes are planned for 1954, but we expect to have two Summer Institutes of Mathematics for Social Scientists in the summer of 1955. At each of these, there will be two curricula, one for those who have studied calculus and one for those who have studied college algebra but not the calculus. It is hoped that detailed announcements will be available in the fall of 1954.

In Sections 2, 3, and 4, Bush, Thrall and Raiffa discuss the following five points:

1. Social scientists as students of mathematics
2. Objectives of the Institute
3. Some considerations in selecting the mathematics to be taught to social scientists
4. The mathematics curriculum of the 1953 Institute
5. Some aspects of teaching mathematics to social scientists

In Section 5 will be found comments.

**2. The development of mathematics courses for social scientists. (Bush).** I will discuss two central problems that arise in developing mathematics programs for social scientists. First I will comment on the ***kinds*** of mathematics which I think should be included in such a program, and second, I will express my views on how such mathematics should be taught.

In planning a mathematics curriculum for social scientists, one can adopt many different criteria for including or excluding any particular topic in mathematics. At one extreme, one might argue that the program should be a survey of *all* kinds of mathematics known to man and let the social scientists of a few

decades hence decide which kinds are most useful to their fields. A person cynical about what has been done in quantitative social science might very well take such an extreme position. But the difficulty with this position is that the program is necessarily too lengthy or too superficial.

Another possible view that is held by some people is that social scientists should learn the kinds of mathematics which are proven tools in physics, chemistry, and astronomy. The trouble with this proposal is that it excludes very little and probably makes a faulty assumption that the social sciences are fundamentally similar to the natural sciences.

An attempt to evaluate what kinds of mathematics are most appropriate for social science problems is another possible approach. But here one can find only a small area of agreement among people who have thought about the problem. The difficulty is simple: to make a sound prognosis about the "future social science," one must have a thorough knowledge of the several social sciences and at the same time know a good deal about all fields of mathematics. No such man exists as far as I know. The people who attempt to draft a program of study on this basis are either competent mathematicians who have had little real experience with social science research, or social scientists who have rather limited mathematical skills but who are anxious to use those skills.

The only satisfactory criterion, I maintain, for selecting content of a program in mathematics for social scientists is the following: Select and weight topics in accordance with what has already been done in applying mathematics to social research. Unfortunately, relatively little has been done in this direction outside of statistics, but the list of papers and books is long enough to allow us to make some inferences. One advantage of this criterion is that the teacher has useful illustrations of the mathematics being taught. This is important, as I will argue later.

On the basis of this criterion I would propose the following outline as desirable for any program of the type we are discussing.

1. *Probability theory with special emphasis on stochastic processes.* The argument for this topic hinges upon the importance of statistics and the growing interest in stochastic models for learning, communication, and small group behavior.
2. *Calculus and differential equations.* Not only is calculus used in most treatments of mathematical statistics, but it also arises in many models for social science phenomena. Some work on finite calculus should be included because of its use in stochastic models and in statistics.
3. *Algebra and axiomatics.* Various uses of matrix algebra, set theory, and axiomatics can be found in the social science literature. I would argue that matrix theory is the most important topic in this category.

The three main categories I have listed should be given about equal weight I believe—and this belief is based upon my acquaintance with what has been done in applying mathematics to social research. I should point out that the list of topics just given is roughly the list of topics included in the program at the

Social Science Research Council's Summer Institute on Mathematics for Social Scientists, held at Dartmouth College last summer.

The other major problem I want to discuss is that of *how* mathematics should be taught to social science students. One part of this problem concerns how much detail and rigor should be included in the program, and this is clearly determined mainly by the time available and the capacities of the students. Concerning this latter matter—the capacity of the social science student for learning mathematical material—let me state some observations. First of all, a great many undergraduates in our universities drift into social science departments because they have had trouble with courses in mathematics and the natural sciences or because they know they dislike such subjects. The clinical psychologist can supply many reasons for such behavior, but prominent among those reasons is unpleasant experiences with mathematics in elementary and secondary schools. Hence, to solve this problem we need to improve secondary education in mathematics or recruit our social science students from a different population. For many years to come, I suspect, we will merely have to accept as a boundary condition that students in the social sciences on the whole do not like mathematics, find it difficult, and indeed have serious psychological blocks against learning mathematics. This fact creates some special problems that do not exist in the teaching of mathematics to engineers, chemists, and physicists.

A closely related observation is that few social scientists will bother with mathematics unless they are highly motivated—have found their lack of mathematical training a serious handicap in their own work or have been convinced by their elders that it will become a serious handicap. Therefore in optional programs or courses in mathematics for social scientists, we can depend on a high level of motivation, a real eagerness to learn, even though these same students have serious psychological problems in this connection. But it is very important, I'm convinced, to keep in mind the source and direction of this strong motivation. In a word, it is directed at finding security in his own field. Consequently, the man who teaches mathematics to social scientists must recognize his dual role of teacher and psychotherapist.

Because of the rather special and intense motivation in the student to acquire skills useful in his own field, the method of teaching is critical. One simply cannot teach mathematics to social scientists of today the way he teaches it to mathematics students; the motivation, the system of needs, the goals of these two kinds of students are entirely different. I think the social science student of mathematics must have frequent reinforcement from manipulative skill at working easy problems—problems he previously could not handle. Such reinforcement will maintain his motivation.

My main point on how mathematics should be taught to social scientists is this. It should be problem-oriented. Each topic in mathematics should be introduced by displaying concrete, non-trivial problems in social research and indicating how some mathematical machinery will be useful. Elegance in mathematical presentation must be sacrificed at times. Social science problems should be

stated, some mathematics taught, and the original problems solved by using that mathematics.

To develop a course such as I propose, a mathematician must necessarily learn a good deal about current problems in the several social sciences, and this is not easy. I have one suggestion: mathematicians should collaborate with social scientists in developing programs and giving courses. By such joint efforts, mathematicians can seriously contribute, I believe, to the development of social science.

**3. The curriculum of the institute. (Thrall).** The 1953 Summer Institute in Mathematics for Social Scientists had four major goals:

A. *Communication.* This is primarily communication with mathematicians; however, experience shows that expressing facts in mathematical language also facilitates communication between social scientists from different areas. One of the bars to communication between mathematician and social scientist is the different meanings attached by them to such words as *relation, function, variable, vector, dimension, continuum.* If the meanings were entirely different the discrepancy would be quickly discovered and cared for; the real difficulty is that the meanings have enough in common to obscure the difference until the damage is done. Since terms such as these are used in social science for essentially mathematical purposes, it seems desirable that the social scientist should learn and then use their mathematical meanings (at least when speaking to mathematicians).

A second bar to communication is lack of technical acquaintance by the social scientist with such basic mathematical concepts as order relation, partial order, real number, continuous functions.

B. *Preparation for Further Courses.* In a single eight-week session one cannot cover all of the mathematics needed by a social scientist. However, a reasonable goal would be to lay a foundation which would enable the student to continue his mathematical education with courses such as advanced calculus, linear algebra, mathematical statistics.

C. *Reading Knowledge.* The amount of mathematics appearing in papers and books in the social sciences is steadily increasing. The mathematical material sometimes appears in appendices and is usually summarized or paraphrased in word form. However, the critical and thorough reader will wish to be able to follow the mathematical arguments. Most of the mathematics used is statistical in nature, but a reading knowledge requires familiarity with such concepts and areas as integral, derivative, order relation, matrix, vector, linear transformation, stochastic process, linear programming, game theory.

D. *Model Building.* The mathematical background needed for model building in the social sciences is much the same qualitatively as for reading knowledge except for a considerably greater emphasis on axiomatics and foundations. Quantitatively, there is a great difference between reading and creating and one cannot expect an eight-week course to produce full-blown experts in the

use of mathematics for model building. However, this goal should be kept in mind in constructing the course.

The goals discussed above would apply to any mathematics course designed for social scientists, but here we are thinking primarily of a coures designed for social scientists who are at or beyond the Ph.D. level when they realize the need for mathematics. At this level it is not practical to undertake the traditional undergraduate mathematics sequence, and even when new undergraduate sequences become available, there will still be a need for special courses to care for mature social scientists. The topics included in the 1953 Summer Institute were selected after consideration of the goals listed above and also keeping in mind the nature of the participants. The topics are not listed in order of importance or in the order in which they should be presented. The letters following each topic indicate the goal (or goals) for which it is important.

1. Set algebra, relations, functions, one-to-one correspondence, equivalence relations, partitions, order relations. A.
2. Axiomatic development of number system including the concept of limit of a sequence. Careful definitions, but proofs limited to heuristic discussions. A, C.
3. Differential and integral calculus. Emphasis on polynomials, but also logarithms and exponential functions. Some analytic geometry included. B, C.
4. Selected topics from advanced calculus, including partial derivatives and multiple integrals. B.
5. Axiomatics, some simple system in detail and the general principles of model building. D.
6. Linear algebra, vector spaces, matrices, linear transformations. B, C.
7. Introduction to probability, sample spaces, stochastic processes, Markov chains. B, C.
8. Models from social science situations; this should include numerous small illustrative examples and also some large scale models such as linear programming, utility theory, learning theory, social choice, game theory, measurement theory. A, C.

**4. Two aspects of a mathematics program for social scientists. (Raiffa).** I would like to divide my comments into two parts: first, the general teaching approach that I would advocate for a program similar to the one given at Dartmouth this past summer; second, the importance of abstract thought (in contradistinction to mathematical technique) to the social scientist.

The points I raise here are, I believe, non-controversial in broad outline—indeed, in broad outline they are trivial. However, the stress given to these issues is more problematical, and since my own viewpoints have changed over the past summer, I would like to outline my subjective opinions at this point.

It is often said that the art of good lecturing is first to say what you are

going to say, say it, and then say what you have said. The price in subject matter covered for following this advice is well worth it *for the social scientist studying mathematics* (for the mathematics student it is not so necessary). In particular, I interpret this advice as follows: Motivation is well worth the expense —this includes motivation of mathematical subject matter, motivations of definitions, motivation of hypotheses, motivation of proofs (rigorous proofs play an important role in such a course—as I see it!)

As regards motivation of subject matter, I think that broad areas should be outlined before plunging into detail. For example, I would not advocate proceeding from a discussion of real numbers to a definition of limit (assuming, of course, that limits were not introduced formally in defining—or should I say in "talking about"—real numbers). I think it would be more appropriate to start off with a series of problems, perhaps related to the social sciences, to abstract these to problems involving maxima and minima, areas, sums of series, *etc.*, and then to show in turn that these have a common abstraction in terms of limits of sequences. Then one could introduce the limit notion, keeping in mind the diverse examples which motivated the abstract subject matter, and using these examples as test cases for the developing theory. Another point I wish to emphasize for the social scientist is that it is a mistake to build up to a principal theorem or application without disclosing our aims during the development of the theory. Of course it might not be possible to point meaningfully to our goal without the necessary machinery at hand, but I think one should not use this as an excuse for postponing the motivation indefinitely.

As regards motivation of definitions, I think it desirable for the student to know where he is going. Tentative definitions should be tried and shown to be faulty. Thus the $\epsilon$-$\delta$ definition of limit should crystallize only after some preliminary fumbling with "definitions" which "really don't capture the idea which we are after." Note that the idea should precede the definition. The non-uniqueness of ways of defining some concept should be indicated.

As regards motivation of hypotheses of a theorem, I think it desirable for the lecturer to check the theorem's validity by means of special cases before discussing its proof. Numerous counter-examples should be given to theorems when hypotheses are altered or relaxed. With this preliminary discussion of the theorem, the student should begin to appreciate its meaning, why certain hypotheses are needed, and what cases are carefully ruled out by the hypotheses. By attempting to find counter-examples to the theorem itself and finding it to no avail the student often gains an insight into the crux of the proof. Incidentally one should not always formulate true theorems; theorems should be thought of as intelligent guesses or conjectures.

As regards motivating the formal proof itself, I wish to point out that understanding each step of a proof is no indication of understanding the proof looked at as a whole. Similarly, if one understands each theorem it does not mean one understands the subject matter as a whole. We should not lose sight of the *Gestalt* by looking too much at details. Learning mathematics by under-

standing in turn each sentence in a formal style of presentation can become a cook-book style of learning. Hence I would stress the "idea" of a proof as much as the formal proof itself, and the "idea" should precede the formal proof. However, one should emphasize that the "idea" falls short of the requirements of a formal proof and these shortcomings should be explicitly pointed out. When the formal proof is completed one should tie in loose ends by checking the places where the hypotheses were needed in the proof. If time is pressing, as it always is, I would much prefer to see the formal proof omitted rather than omitting the motivating remarks and/or the "idea" of the proof (especially since the formal proof can be found in texts while the motivation is not often enough in print). However, if this program is too time-consuming for the lecturer, part of it could be assigned explicitly as a problem for the student.

In the realm of motivation one should stress the creativity of the mathematician and one should indulge in heuristic arguments. It should be pointed out that mathematical creativity is largely a trial-and-error procedure (similar to empirical research) and that the formal proof is usually quite different from the pioneering attack on the problem.

The technique of starting a lecture by summarizing the pertinent information from previous lectures helps the student to see the main trend, but what I consider more important is that it affords an opportunity for the lecturer to repeat the material in a more symbolic and abstract form, unencumbered by motivation and examples, and thus far more compact; and then there is the secondary effect of having the student aware of all the innuendos and the rich meaning involved in a pithy abstract mathematical statement. Hence I would recommend repeated summaries of material covered in order to raise the mathematical maturity level of the students.

I concur wholeheartedly with Bush that, psychologically speaking, the student has to have some sense of accomplishment and should not become overawed with the extensive scope of the material. To this end, there should be numerous graded exercises. We should play along with the game that if one can manipulate and substitute in formulas then one "understands" the theory. We should not destroy this false sense of security. Where one should draw the line is hard to say, but personally, after the summer's experience, I would say that there should be more manipulation than we gave this summer. Any program such as the Summer Institute must face the problem of weighing the emphasis between abstract and concrete presentation of the subject matter (*e.g.*, vectors *vs.* $n$-tuples, linear transformations *vs.* matrices, stress on limit notions *vs.* more topics in maximization, *etc.*). If approach A is easier to get across than approach B, is it better? Not necessarily. I feel that another indicator that one should take into account is whether A or B is more conducive to the general "mathematical maturity level" of the student, and I would go so far as to introduce material which is primarily intended to increase mathematical maturity. Of course, I admit that I do not know how to do this effectively, and I do not deny that solving

a series of differential equations may contribute to the mathematical maturity of the student.

In order to defend retaining some moderate degree of abstract presentation in programs of this kind for the social scientist, I would like to say something about axiomatics in general. I choose this topic because I think it is defended too often for reasons which I do not consider important; but, on the other hand, I think it is important to include this topic in our program. Some people argue that for approaches to the social sciences to be sophisticated they must be axiomatic in nature; they point to game theory as an example. On the contrary, I think that premature emphasis on the axiomatic development of a subject matter—learning theory, for example—can be detrimental. If applied to large areas it can cause sterility of mathematical ideas. The fact that Von Neumann axiomatized game theory is not the reason that game theory is interesting to some mathematicians. The reason is rather that the theory presents well-formulated unsolved problems which would exist regardless of the axiomatic formulation of the subject matter. Certainly I would agree that it is sometimes beneficial for the social scientist to try to axiomatize an area—mainly because this directs his attention to the structure and to some of the basic notions of the area. However, axiomatic-type thinking has played and will play in the future an increasingly important role in mathematical work in the social sciences. Often in the behavioral sciences, when one grapples with some nebulous material (*e.g.*, cohesiveness of a group, so-called "rational" behavior, "socially desirable" welfare functions, *etc.*), definition after definition is discarded or discredited because it does not fit the bill in certain situations (*i.e.*, does not fulfill the hazy desiderata one has formed in one's own intuition). It then behooves one to pay more attention to these subjective desiderata and to formulate conditions one wishes the definitions to satisfy. These conditions should be viewed as axioms, and one should then determine whether they are consistent, independent, and categorical. If a set of mutually inconsistent conditions is prescribed, then one must review one's intuition. As a case in point, Arrow, in his *Social Choice and Individual Value*, formulates some historically important and intuitively palatable conditions for a social welfare function to satisfy, only to show that these conditions are inconsistent. This plays the valuable role of directing our attention to our inconsistent intuitions rather than to the shortcomings of specific proposals. A similar example can be pointed out in the area of decision making under uncertainty, where a set of reasonable but inconsistent conditions has been given (Milnor, Chernoff) for a rule of inductive behavior. These conditions are then studied further to show that various subsets lead to various principles (minimax principle of Wald, equally likely *a priori* principle of Laplace, *etc.*). Thus the investigations of these conditions serve to resolve certain philosophical difficulties for oneself. Another area in which there has been much thinking in axiomatic terms is in the solution concept of the $n$-person game. If a set of conditions or requirements turns out to be consistent and categorical in the sense that it uniquely characterizes a concept, then this procedure

implicitly serves as a definition of that concept; if a set of conditions turns out to be consistent but permits a subset of possible interpretations then one has localized one's consideration to this well-defined subset and one hopes that various arguments will hold for all the interpretations of that subset (*i.e.*, have an argument remain invariant over the subset characterized by our initial desiderata). Instances of these variants have turned up in the social sciences, and they are becoming increasingly popular. I thus conclude that the study of axiomatics in general is pertinent to the study of mathematics for social scientists.

**5. Discussion. (Madow).** On the whole, the curriculum of the 1953 Summer Institute was chosen in the way outlined by Bush, and, indeed, this is reflected by the rather close agreement between the list of topics recommended by Bush and that used in the 1953 Institute.

I do believe that it is an exaggeration to say that one should select and weight topics in accordance with what has already been done in applying mathematics to social research. We all know that certain lines of approach are developed and then an impass reached. For that reason, in reviewing what has been done, one must also take into account what is being done and what is likely to be done in the future. Thus, twenty years ago, one would have said that it was quite unnecessary to study algebra and set theory in economics and statistics. Just about that time, the importance of both of these subjects was beginning to be recognized and today the topics are far more important than they were twenty years ago. On the other hand, the importance of the calculus, particularly in statistics, has been sharply reduced. Thus, as in so many other fields, merely to consider the past would result in a biased program.

On the other hand, to consider only one's own personal research or the research activities of a small group would be likely to be very much biased also. It is a question of judgment and one can not avoid some forecast.

A similar comment applies to Bush's statement that he could argue that matrix theory is the most important topic among matrix algebra, set theory, and axiomatics. All three are needed, and the precise balance will depend on the individuals and needs concerned.

On the whole, I agree with Bush's characterization of social scientists, namely, that if they study mathematics, they will turn out to be highly motivated but also that many will have had some bad experiences with mathematics. A further difficulty faced by the social scientist as an undergraduate, when he starts studying mathematics, will be that he will not find the mathematics used in courses in his own subject matter. This is a disadvantage not faced by the physical scientist or engineer who, after studying some mathematics finds that it is referred to in other courses. Thus, the social scientist comes with high motivation but the knowledge that he has had previous difficulties and the feeling that he could probably get along without the mathematics. Then, too, he is not part of a captive audience, as are the students from the physical sciences

and engineering. The result is that he definitely requires a higher level of teaching than do the usual students of mathematics courses. In this connection, it is important to realize that social scientists believe that learning is better advanced through reinforcement and reward than through challenge, whereas in our mathematics courses, we tend to challenge the student. Here, again, we need better mathematics teaching. It is worth noting that the great interest of mathematicians in the type of curriculum we suggest may well produce a higher level of teaching than now exists.

Bush's discussion of how to teach mathematics to social scientists should be read in conjunction with the comments of Raiffa on the same subject. Perhaps I can summarize the situation by saying that we think that mathematics should be taught, not presented.

In our announcement of the Institute, we listed as the goals in order: model building, reading knowledge, and preparation for further courses. It was our feeling that all but the very poorest students would be able to communicate, that many of the students would be able to take further courses and read mathematical literature in their own fields, and that a few would be able to create models. Of the various goals listed by Thrall, I would say that the most of our students and the faculty were in agreement that the most important single result would be an increase in reading knowledge, and that this was attainable during the eight-week session.

Although the course we gave at Dartmouth was designed primarily for mature students, we believe that with very little change it would be a more suitable mathematics course for undergraduate social scientists than one or two years of the classical undergraduate curriculum in mathematics. I even think that the student would benefit far more from taking this course one year and repeating it the second than he would from the current first two years of college mathematics. Of course, there are administrative reasons why this may not be practicable, but I mention it to stress the point of view.

I think that there are many differences of opinion concerning Thrall's identification of mathematical topics and goals. To mention just one, to the economist and to the statistician, the reasons for studying the partial derivative far transcend the preparation for further courses in mathematics. But the precise identifications are not essential.

Whether the axiomatic development of the number system was needed is doubtful. If needed, it is hard to see why it is important for the goal of communication. It should not be understood that each of the topics by Thrall has equal importance. Thus, far more space would be given to item 6 and to item 7 than, let us say, to item 2.

All of us are in general agreement with the distribution of time mentioned by Bush. The only differences that would occur would be of the order of replacing one-third by one-quarter, or by one-fifth, neither of which would be great changes.

I should like to stress my agreement with Raiffa's comments on the teaching of mathematics, and to add that they really go beyond the teaching of mathematics to social scientists, and apply to teaching of mathematics in general. This is particularly true with respect to his statement that understanding each step of a proof is no indication of understanding the proof looked at as a whole, and that understanding each theorem does not mean that one understands the subject matter as a whole. Of course, it is not easy to achieve the full understanding, but this is a challenge to those who teach mathematics, as well as to those who study it.

It seems likely that axiomatic *versus* non-axiomatic treatments will continue to be a source of controversy. In general, social scientists will not find axiomatic approaches too difficult. Considerable gains to them will result from the consciousness that comes from the axiomatic approach, of the way in which mathematics and the world of experience interact.

After a lapse of many centuries, axiomatic approaches have again become important in many parts of mathematics. Here, as in many other questions of education, we must be certain that our preferences, be they for axiomatic or non-axiomatic approaches, are based on rational reasons rather than on what we happen to have studied.

I have just two more comments. At the present time, when a student has had undergraduate mathematics and decides to continue the study of mathematics beyond the junior year, he often feels as though the subject matter he is studying has changed completely. With the curriculum that we taught in the summer of 1953, this would not be true. I think, therefore, that that curriculum would be superior to the present undergraduate curriculum not only for the social scientist but also for students of mathematics, and indeed, for students from all fields other than those such as the physical sciences and engineering in which it is important that certain technical processes be known at certain times. Even for such students, I think that a curriculum such as we are discussing would be better than the present curriculum, but this needs to be worked out more.

Doubtless, there will be many questions raised concerning the inclusion or exclusion of certain topics. For example, if differential equations are taught, should one include differential equations of the second order? I suggest that we must avoid this type of question at present. The mathematics that is used in social science papers depends on the mathematical background that the researcher happens to have had. It would be impossible to cover all points, and it will be impossible to satisfy all people on the amount of time devoted to different topics. What we need is to have more courses, more experiments. Then, gradually, a curriculum will evolve.

## MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

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### A NOTE CONCERNING THE QUOTIENT $(r^{p-1}-1)/p$

C. A. NICOL, University of Texas

This note contains two theorems concerning the Fermat Quotient, and a theorem concerning a sum of powers of primitive roots modulo an odd prime. The symbol $q(r)$ will denote the quotient $(r^{p-1}-1)/p$.

THEOREM 1. *If $p$ and $r$ are primes and $r$ is a primitive root modulo $p$, then*

$$q(r) = (r-1)\{e(r) + (p-1)/2\} \tag{1}$$

*where $e(r)$ denotes the exponent of the highest power of $r$ which divides $q(r)!$.*

*Proof:* Consider first of all the identity

$$(r^{p-1}-1)/p = (r-1)\{(r^{p-2}-1)/p + \cdots + (r-1)/p + (p-1)/p\}. \tag{2}$$

For each integral value of $\alpha$ in the interval $1 \leqq \alpha \leqq p-2$ the quotient $(r^{\alpha}-1)/p$ may be written as $I_{\alpha}+s_{\alpha}/p$ where $I_{\alpha}$ is a non-negative integer and the quotient is a proper fraction. Also, the numbers $s_1, s_2, \cdots, s_{p-2}$ are equal to the set $1, 2, \cdots, p-2$ in some order. Thus the identity may be written as

$$(r^{p-1}-1)/p = (r-1)\{I_{p-2} + \cdots + I_1 + (p-1)/2\}.$$

If $x$ is a real number let the symbol $[x]$ denote the largest integer not exceeding $x$. If $c$ is an integer in the interval $1 \leqq c \leqq p-2$ we wish to establish the following relation:

$$[(r^{p-1}-1)/pr^c] = [(r^{p-c-1}-1)/p] = I_{p-c-1}. \tag{3}$$

The following known lemmas are employed [1].

LEMMA 1. *If $x$ is a non-negative real number, and if $a$ is a rational integer, then*

$$[x/a] = [[x]/a].$$

LEMMA 2. *If $n$ is a positive integer and $p$ is a prime, then the exponent of the highest power of $p$ which divides $n!$ is equal to*

$$[n/p] + [n/p^2] + [n/p^3] + \cdots.$$

If $x$ in Lemma 1 is replaced by $(r^{p-1}-1)/r^c$, relation (3) follows. Furthermore if the quotient $(r^{p-1}-1)/pr^c$ is written as $((r^{p-1}-1)/p)/r^c$ and if $n$ in Lemma 2 is replaced by the numerator of the latter quotient, the theorem is immediate.

THEOREM 2. *If $S(m^k)$ denotes the sum of the k-th powers of the primitive roots modulo an odd prime p, then*

$$S(m^{p-1}) = \{p(p-1)/2\}S(m) + p\sum_{i=1}^{N}{}' (m_i - 1)D(m_i) - N(p+1)(p-2)/2,$$

*where N is the number of primitive roots modulo p and* $D(m_i) = [q(m_i)/m_i] + [q(m_i)/m_i^2] + \cdots$.

*Proof.* Since a primitive root need not be a prime we write relation (1) in the form,

$$m^{p-1} = 1 + pD(m)(m-1) + (m-1)p(p-1)/2$$

where

$$D(m) = [q(m)/m] + [q(m)/m^2] + [q(m)/m^3] + \cdots.$$

Summation over the primitive roots mod $p$ yields the theorem.

The following theorem is not a generalization of Theorem 1.

THEOREM 3. *If p and r are primes and r belongs to d modulo p, then*

$$q(r) = (r-1)\{e(r) - (p-1)e'(r)/d\} + (p-1)(r^d - 1)/pd$$

*where $e'(r)$ denotes the exponent of the highest power of r dividing $((r^d-1)/p)!$ and $e(r)$ is defined in theorem* 1.

*Proof.* If $d=1$ or $p-1$ the theorem is trivial. Hence suppose that $1<d<p-1$. In relation (2) consider the quotients $(r^\alpha-1)/p = I_\alpha + s_\alpha/p$. We may write $r^\alpha - 1 \equiv s_\alpha \pmod p$. Then we have $s_\alpha \equiv s_\beta \pmod p$ if and only if $\alpha \equiv \beta \pmod d$ and, since $0 \leqq s_\alpha \leqq p-1$, $0 \leqq s_\beta \leqq p-1$, it follows that $s_\alpha = s_\beta$ if and only if $\alpha \equiv \beta \pmod d$. Also since $d$ divides $(p-1)$ we have $(p-1)/d$ integers $s_j$, $j=1, \cdots, d-1$, and then equation (2) becomes

$$(4)\quad q(r) = (r-1)\{I_{p-2} + \cdots + I_1 + (p-1)(s_1 + \cdots + s_{d-1})/pd + (p-1)/p\}.$$

By the argument given in Theorem 1, the sum $I_{p-2} + \cdots + I_1$ is equal to $e(r)$. Also $(s_t)/p = (r^t-1)/p - [(r^t-1)/p]$ for $1 \leqq t \leqq (d-1)$. Hence the sum $(s_1 + \cdots + s_{d-1})/p$ is equal to $(r^d-1)/(r-1)p - d/p - e'(r)$ since from the argument in Theorem 1 we have

$$\sum_{t=1}^{d-1} (r^t - 1)/p = e'(r).$$

When these quantities are substituted in equation (4) the theorem is immediate.

**Bibliography**

I. Introduction to Number Theory, T. Nagell, 47, Uppsala, 1951.

## CYCLOTOMY AND THE CONVERSE OF FERMAT'S THEOREM

MORGAN WARD, California Institute of Technology

The following theorem of A. Hurwitz [1] was stated without proof in answer to a question raised by E. B. Escott [2] regarding a test for the primality of the Fermat numbers $2^k+1$ ($k$ a power of two) stated without proof by E. Lucas in his *Theorie des Nombres* [3].

HURWITZ'S THEOREM: *Let $Q_n(x)$ denote the cyclotomic polynomial of order $n$ and degree $\phi(n)$. Then $n$ is a prime number if there exists an integer $a$ such that*

$$Q_{n-1}(a) \equiv 0 \pmod{n}. \tag{1}$$

This theorem is a simple consequence of the converse of Fermat's theorem. For let $n$ be greater than one. Then the factorization of $x^{n-1}-1$ into its irreducible factors over the rational field is

$$x^{n-1} - 1 = \prod_{d \mid n-1} Q_d(x). \tag{2}$$

Here the product extends over all distinct divisors $d$ of $n-1$.

The discriminant $\pm(n-1)^{n-1}$ of $x^{n-1}-1$ is prime to $n$. Consequently if $d$ and $d'$ are distinct divisors of $n-1$, the resultant of $Q_d(x)$ and $Q_{d'}(x)$ is always prime to $n$. Hence for any integer $a$ and any proper divisor $d$ of $n-1$, the three numbers $Q_{n-1}(a)$, $Q_d(a)$ and $n$ are co-prime.

Now assume that the congruence (1) is satisfied. By what precedes, $Q_d(a)$ is prime to $n$ for every proper divisor $d$ of $n-1$. But by (2)

$$a^{n-1} - 1 \equiv \prod_{k \mid n-1} Q_k(a) \equiv 0 \pmod{n},$$

$$a^d - 1 \equiv \prod_{k \mid d} Q_k(a) \not\equiv 0 \pmod{n}, \qquad d \mid n-1;\ d < n-1.$$

Therefore $n$ is a prime by the converse of Fermat's theorem. (For this converse see for example, O. Ore, *Number Theory*, Chapter XIV.)

For example, $Q_2(a) = a+1 \equiv 0 \pmod 3$ for $a=2$; $Q_6(a) = a^2-a+1 \equiv 0 \pmod 7$ for $a=3$. Hence 3 and 7 are primes. But the most interesting application of the theorem is that made by Hurwitz himself to the case when $n=2^k+1$ so that $Q_{n-1}(x) = x^{(n-1)/2}+1$; namely $n=2^k+1$ is a prime number if and only if there exists an integer $a$ such that $a^{(n-1)/2} \equiv -1 \pmod{n}$. For $a=3$, this result is the test quoted by Lucas and used extensively by him and by other arithmeticians for investigating particular Fermat numbers.

### References

1. A. Hurwitz, Mathematische Werke, vol. 2, page 747.
2. Intermédiare des Math., vol. II, 1896, pp. 80 and 214.
3. Lucas published proofs elsewhere, but the test is due to T. Pepin. For a history and references to other proofs see Dickson, *History of the Theory of Numbers*, vol. 1, Chap. XV.

## CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

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### TWO ELEMENTARY FALLACIES

EDWIN HALFAR, University of Nebraska

Because of the similarity to arguments to which students in elementary classes are frequently exposed, the following elementary fallacies seem to have especial appeal.

1. The hyperbola and parabola are identical. The locus of points which are such that the slope of the line joining any one of them to the fixed point $(a, 0)$ is equal to the point's distance from the $y$-axis is $x^2-ax=y$. On the other hand, if the fixed line is $ax+by+c=0$ and the fixed point $(x_1, y_1)$, then the locus is $(ax+by+c)/(a^2+b^2)^{1/2}=(y-y_1)/(x-x_1)$, a hyperbola. Since, as every analytic geometry student knows, by a translation and a rotation the line $ax+by+c=0$ and the point $(x_1, y_1)$ can be transformed into the $y$-axis and $(a, 0)$ respectively; and furthermore since the conic sections are invariant under translation and rotation, the parabola is the same as the hyperbola.

2. The length of the ellipse is $\pi(a+b)$. Given the ellipse with semi-axes $a$ and $b$ respectively, increase the axes by $\epsilon$ on each end. The difference between the areas enclosed by the two ellipses thus formed is $\pi(a+\epsilon)(b+\epsilon)-\pi ab=L'\epsilon$ where $L'$ is a mean line lying between the two ellipses. After an algebraic simplification, one has $\pi[(a+b)+\epsilon]=L'$. Passing to the limit as $\epsilon$ tends to zero, one has $\pi(a+b)=L$, the length of the ellipse, since $L'$ tends to $L$ as $\epsilon$ tends to zero.

As a "check" of the "validity" of this result, one may observe that the circle is a special case of the ellipse so that by letting $a=b$, one obtains $L=2\pi a$.

### ON A GRAPHICAL SOLUTION OF THE FIRST ORDER LINEAR DIFFERENTIAL EQUATION

M. S. KLAMKIN, Polytechnic Institute of Brooklyn

The linear differential equation

$$\frac{dy}{dx}+P(x)y=Q(x) \tag{1}$$

can always be solved by quadratures to

$$y=e^{-\int P\,dx}\int Qe^{\int P\,dx}dx. \tag{2}$$

However, in many cases the quadratures cannot be effected in finite form. For these cases we can use the following graphical method.

One first plots the curves $C_1$: $[y=F(x)]$, and $C_2$: $[x=G(y)]$, (assumed to be continuous), as shown in Figure 1. Through an arbitrary point $P(x, y)$, a line is drawn parallel to the $y$-axis intersecting $C_1$ at $A$; through $A$, a line is drawn parallel to the $x$-axis intersecting $C_2$ at $B$. One then draws the line $PB$. Since the coordinates of points $A$ and $B$ are $(x, F(x))$, and $(GF(x), F(x))$, respectively, the slope of $PB$ is $(y-F(x))/(x-GF(x))$ $[GF(x)$ is to mean $G(F(x))]$. A small section of the integral curve to

$$\frac{dy}{dx} = \frac{y - F(x)}{x - GF(x)}, \tag{3}$$

passing through $P$ is given by a small segment $PP_1$ of $PB$. The procedure is now repeated starting from $P_1$. The proximity of the successive points, $P$, $P_1$, $P_2$, $\cdots$, taken will determine the accuracy of the construction.

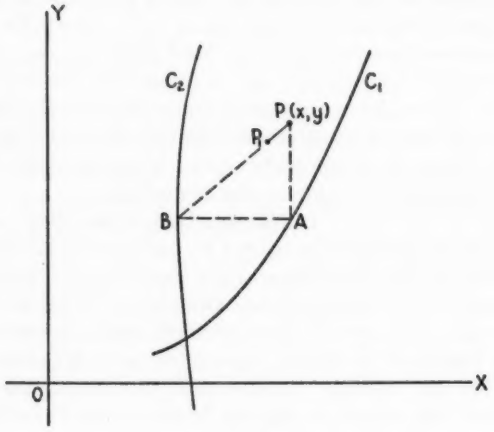


FIG. 1

Now, we have to identify the functions $F(x)$ and $G(x)$. By comparing equations (1) and (3) we must have

$$P(x) = \frac{1}{GF(x) - x}, \tag{4}$$

$$Q(x) = \frac{F(x)}{GF(x) - x}. \tag{5}$$

Solving equations (4) and (5) simultaneously, we get

$$F(x) = \frac{Q(x)}{P(x)}, \tag{6}$$

$$GF(x) = \frac{1}{P(x)} + x, \quad \text{or} \tag{7}$$

$$G(y) = \frac{1}{PF^{-1}(y)} + F^{-1}(y). \tag{8}$$

The above method can also be adapted to solving the Bernouilli equation.

Using a similar procedure, but now plotting four curves instead of two, we can graphically solve the non-linear differential equation:

$$\frac{dy}{dx} = \frac{\phi(x) - G(y)}{\psi(x) - F(y)}. \tag{9}$$

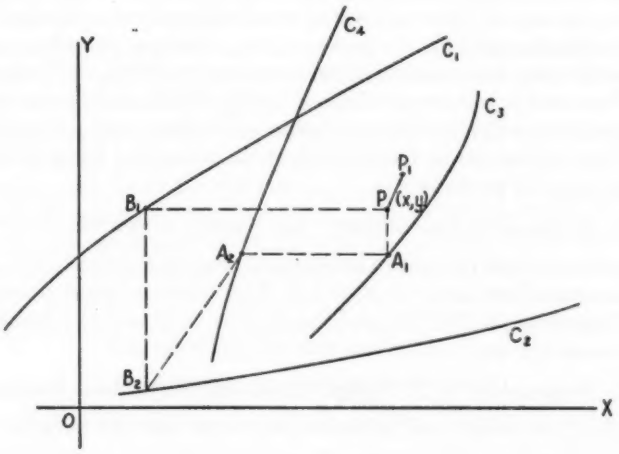


FIG. 2

In Figure 2, the equations of the curves $C_1$, $C_2$, $C_3$, and $C_4$ (assumed to be continuous) are given by

$$\begin{aligned} C_1&: \quad x = F(y), \\ C_2&: \quad y = GF^{-1}(x) \\ C_3&: \quad y = \phi(x), \\ C_4&: \quad x = \psi\phi^{-1}(y). \end{aligned} \tag{10}$$

$PB_1$, $A_1A_2$, $B_1B_2$, and $PA_1$ are drawn parallel to the coordinate axes, while $PP_1$ is drawn parallel to $A_2B_2$.

It follows from the construction that the slope of $PP_1$ satisfies equation (9).

This latter construction generalizes the procedure of J. P. Russell and the author (this MONTHLY, vol. 61, 1954, pp. 188–189).

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTION

E 1131. *Proposed by L. A. Ringenberg, Eastern Illinois State College*

A student solves the differential equation $(dy/dx)^2=x^2$ and reports as follows: "General solution $(2y-2c)^2=x^4$; $p$-discriminant $x=0$; $c$-discriminant $x=0$. Since the $p$-discriminant locus consists of the envelope, cuspidal, and tac loci while the $c$-discriminant consists of the envelope, cuspidal, and nodal loci, and since the line $x=0$ is not an envelope, it ought to be a cuspidal locus. But the general solution consists of the parabolas $y=x^2/2+c$ and $y=-x^2/2+c$, and parabolas don't have cusps. How can there be a cuspidal locus if there are no cusps?" Resolve the predicament.

E 1132. *Proposed by Leon Bankoff, Los Angeles, California*

A common external tangent of two circles, tangent externally at $C$, cuts their smallest circumscribed circle in $P$ and $Q$. The common internal tangent at $C$ intersects the minor arc $PQ$ in $E$, and the major arc $QP$ in $F$. $PC$ extended meets the outer circumference in $K$. Show that arc $QK$ = arc $KF$.

E 1133. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute*

Show that a necessary and sufficient condition that the imaginary roots of

$$f(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n = 0, \qquad a_0 \neq 0,$$

occur in conjugate pairs is that the numbers $a_0, a_1, \cdots, a_n$ all lie on a common ray emanating from the origin in the complex plane.

E 1134. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that a square integer is not a perfect number.

E 1135. *Proposed by Vern Hoggatt, San Jose State College*

If a third order determinant has elements 1, 2, 3, $\cdots$, 9, what is the maximum value it may have?

### SOLUTIONS

#### An Optimum Problem

E 804 [1948, 96]. *Proposed by S. H. Gould, Purdue University*

Denote by $U$ the ellipsoid $a_1^2x_1^2+a_2^2x_2^2+a_3^2x_3^2=1$, by $E_b$ the ellipse of intersec-

tion of $U$ with the plane $b_1x_1+b_2x_2+b_3x_3=0$, by $(p_1, p_2, p_3)$ a point variable on $E_b$, and by $E_p$ the ellipse of intersection of $U$ with the plane $p_1x_1+p_2x_2+p_3x_3=0$.

Determine $(p_1, p_2, p_3)$ so as to minimize the major axis of $E_p$.

*Solution by R. Beil and the Proposer, Purdue University.* Choose $b_1, b_2, b_3$ so that the endpoint of the vector $b=(b_1, b_2, b_3)$ lies on $U$, and consider the two cases: (i) $b$ is shorter than the mean axis, call it $u^{(2)}$, of $U$, (ii) $b$ is not shorter than $u^{(2)}$.

(i) Take $p=(p_1, p_2, p_3)$ orthogonal to $b$ and $u^{(2)}$. Then $u^{(2)}$ is a principal semi-axis of $E_p$, since it is orthogonal to the tangent at its endpoint. But $u^{(2)}$ must be the major axis of $E_p$, since the shorter vector $b$ is also a semi-diameter of $E_p$. So $p$ solves the problem, since $u^{(2)}$ is the shortest possible major axis for all ellipses cut off from $U$ by a plane through the center.

(ii) Denote by $B$ the plane $b_1x_1+b_2x_2+b_3x_3=0$, by $T$ the tangent plane to $U$ at the endpoint of $b$, and by $S$ the plane through the center parallel to $T$. Take $p$ along the intersection of $S$ and $B$. Then $b$ is a principal semiaxis of $E_p$, since it is orthogonal to the tangent at its endpoint. But $b$ must be the major axis of $E_p$, since no minor axis can be longer than $u^{(2)}$. So $p$ solves the problem, since $b$ is a semi-diameter for every possible $E_p$.

If $a_1^2 \leqq a_2^2 \leqq a_3^2$, we readily calculate: (i) $p_1, p_2, p_3$ are proportional to $b_3, 0, -b_1$, (ii) $p_1, p_2, p_3$ are proportional to $b_2b_3(a_2^2-a_3^2)$, $b_3b_1(a_3^2-a_1^2)$, $b_1b_2(a_1^2-a_2^2)$.

This solution is a special case of results obtained for Hilbert space by H. F. Weinberger. See his article, "An optimum problem in the Weinstein method for eigenvalues," *Pacific Journal of Mathematics*, vol. II (1952), p. 413.

### Generalization of the Inequality of Bernoulli

E 1101 [1954, 123]. *Proposed by M. S. Webster, Purdue University*

The inequality of Bernoulli is often stated as follows: If $h>-1$ and $h\neq 0$, then $(1+h)^n>1+nh$, where $n$ is an integer greater than unity. What is the generalization if $n$ is an arbitrary real number?

*Solution by A. R. Hyde, West Hartford, Conn.* Clearly, the line $g(h)=1+nh$ is tangent to the curve $f(h)=(1+h)^n$ at (0, 1). The algebraic sign of $f''(h)=n(n-1)(1+h)^{n-2}$ shows by the direction of curvature that $f(h)<g(h)$ in the range $0<n<1$, $f(h)=g(h)$ for $n=0$ and $n=1$, and $f(h)>g(h)$ if $n<0$ or $n>1$.

Also solved by M. S. Klamkin, R. Z. Vause, and the proposer.

Klamkin pointed out that the Bernoulli inequality is contained in the following more general one, found on p. 37 of *Inequalities* by Hardy, Littlewood, and Pólya: If $x>0$, $0<m<n$, then

$$(1 + x/m)^m < (1 + x/n)^n.$$

The Bernoulli inequality is obtained by setting $x=nh$ and $m=1$.

**Commutatively Factorable Matrices**

E 1102 [1954, 123]. *Proposed by Harley Flanders, University of California*

Let $C$ be an $n \times n$ matrix such that whenever $C=AB$ then $C=BA$. What is $C$?

*Solution by the Proposer.* We first prove that $C$ commutes with every $n \times n$ matrix, $A$. If $A$ is non-singular we set $B=A^{-1}C$, so that $AB=C$. Then $BA=C$, so that $CA=(AB)A=A(BA)=AC$. If $A$ is singular then, for a suitable $x$, matrix $A_1=xI+A$ is non-singular, $A_1$ commutes with $C$, and hence $A$ commutes with $C$. Now we use the fact that the only matrices $C$ which commute with every matrix are the scalar matrices $C=cI$. The constant $c$ cannot be zero unless $n=1$, for

$$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = 0, \quad \text{but} \quad \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

Conversely, if $C=cI$, with $c$ different from zero, then $C=AB$ implies $C=BA$.

Also solved by Joel Brenner, G. E. Forsythe, D. S. Greenstein, Marvin Marcus, D. C. B. Marsh, L. L. Pennisi, P. P. Saworotnow, and Olga Taussky Todd.

**An Ellipsoidal Locus**

E 1103 [1954, 123]. *Proposed by Paul Monsky, Brooklyn, N. Y.*

Find the locus of the vertex of a tri-rectangular trihedral angle which moves so that its edges intersect a given circle.

*Solution by Martinus Esser, Georgia Institute of Technology.* Let the equations of the circle be $x^2+y^2=r^2$, $z=0$. Let the vertex of the trihedral be $V$ (coordinates $X$, $Y$, $Z$), and let the intersections of the edges with the circle be $A$, $B$, $C$. The four vertices $V$, $A$, $B$, $C$ determine a rectangular parallelepiped, with three additional vertices in the plane $z=-Z$ and with eighth vertex in the plane $z=-2Z$. The coordinates of the center $\Omega$ of this parallelepiped are $0$, $0$, $-Z/2$. Equating the distances $|\Omega V|$ and $|\Omega A|$, we obtain the desired equation $X^2+Y^2+2Z^2=r^2$ of the locus of $V$. The locus is thus an oblate spheroid containing the given circle.

Also solved by Michael Goldberg, C. S. Ogilvy, Roscoe Woods, and the proposer.

**A Constrained Function**

E 1104 [1954, 124]. *Proposed by J. R. Hatcher, Fisk University*

Let $g(x)$ be a given continuous function satisfying $\int_\alpha^\beta g(x)dx=0$. Find $f(x)$ such that $\int_\alpha^\beta f^2(x)dx$ is a minimum subject to the conditions $\int_\alpha^\beta f(x)dx=\gamma$ and $\int_\alpha^\beta f(x)g(x)dx=\delta$, where $\alpha$, $\beta$, $\gamma$, $\delta$ are constants.

*Solution by Chih-yi Wang, University of Minnesota.* The relation

$$\int_\alpha^\beta [f(x) - \lambda g(x) - \mu]^2 dx \geqq 0, \qquad \lambda, \mu \text{ real},$$

implies that

$$\int_\alpha^\beta f^2(x)dx \geqq \delta^2/K + \gamma^2/(\beta - \alpha) - (K\lambda - \delta)^2/K - [(\beta - \alpha)\mu - \gamma]^2/(\beta - \alpha),$$

where

$$K = \int_\alpha^\beta g^2(x)dx.$$

It is obvious that the required minimum value $\delta^2/K+\gamma^2/(\beta-\alpha)$ is attained if and only if $\lambda=\delta/K$ and $\mu=\gamma/(\beta-\alpha)$, whence

$$f(x) = \delta g(x)/K + \gamma/(\beta - \alpha).$$

Also solved by D. W. Allan, M. S. Klamkin, and the proposer.

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## ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.*

### PROBLEMS FOR SOLUTION

4603. *Proposed by H. S. Shapiro, New York University*

Given $x_i \geqq 0$, $i=1, 2, \cdots, n$. Establish

$$\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \cdots + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2} \geqq \frac{n}{2},$$

equality occurring only if all denominators are equal.

4604. *Proposed by Richard Bellman, The Rand Corporation, Santa Monica, California*

Suppose it is necessary to traverse a distance $x$ where the legal speed is $v_0$. If you travel at a speed $v$, greater than $v_0$, there is a probability, $p(v)dt$, of being

stopped in the time interval $t$ to $t+dt$ (and given a ticket). This consumes a fixed time $r$. (Actually, $r$ should be a function of $v$.) Let us also assume that $p(v_s)=1$, for $v \geqq v_s$, the suicidal velocity. At what speed should one travel to minimize the expected time required to cover the distance $x$?

4605. *Proposed by D. J. Newman and W. E. Weissblum, Republic Aviation Corporation, Farmingdale, N. Y.*

Given an open, unbounded set of positive reals. Prove that there exists a real number such that infinitely many integral multiples of it lie in the set.

4606. *Proposed by H. R. Smith, McCoy College, Baltimore, Maryland*

Given a set of $n$ countries, $a_1, a_2, \cdots, a_n$, prove that there exist no more than $3n-6$ distinct boundaries of the form $mn$ between the countries $a_m$ and $a_n$. By "distinct" is meant that if $rs$ and $mn$ are two such boundaries and $a_r$ is the same as $a_m$ then $a_s$ is not the same as $a_n$.

4607. *Proposed by H. S. M. Coxeter, University of Toronto*
Evaluate

$$\frac{1 - 2^{-2} + 4^{-2} - 5^{-2} + 7^{-2} - 8^{-2} + \cdots}{1 + 2^{-2} - 4^{-2} - 5^{-2} + 7^{-2} + 8^{-2} - \cdots}.$$

## SOLUTIONS

### Limit of a Sum

4539 [1953, 336]. *Proposed by J. Gallego-Diaz, Madrid, Spain*

Determine

$$\lim_{n\to\infty} [\tanh 1 + \tanh 2 + \cdots + \tanh n - \log \cosh n].$$

*Solution by H. A. Robinson, Agnes Scott College.* Let

$$S(n) = \tanh 1 + \tanh 2 + \cdots + \tanh n - \log \cosh n.$$

Now

$$\log \cosh n = n + \log (1 + e^{-2n}) - \log 2,$$
$$\tanh k = 1 - 2(e^{2k} + 1)^{-1}$$

imply

$$S(n) = \log 2 - 2\sum_{j=1}^{n} \frac{1}{e^{2j} + 1} - \log (1 + e^{-2n}).$$

As $n\to\infty$ the second item of $\lim S(n)$ is a known convergent series whose value is easily computed as $0.2801\cdots$, and the last item approaches zero. Hence $\lim S(n) = \log 2 - 0.2801 = 0.413$ approximately.

Also solved by W. E. Briggs, Hermann von Schelling, and O. E. Stanaitis.

**The Möbius Function and a Certain Divisor Function**

4541 [1953, 336]. *Proposed by Jack Warga, Republic Aviation Corporation, Farmingdale, N. Y.*

Let $d_k(n)$ be defined for integers $k \geqq -1$ and $n \geqq 1$ by:

$$d_{-1}(n) = \delta_{1n} = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1, \end{cases}$$

$$d_{k-1}(n) = \sum_{r_1 r_2 \cdots r_k = n} 1 = \text{the number of}$$

representations of $n$ as a product of $k$ factors, for $k \geqq 1$.

Let $\mu(j)$ be the Möbius function.

Let $\lambda(j) = \alpha_1 + \alpha_2 + \cdots$ for $j = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$, (the decomposition of $j$ into prime powers).

Prove that, for all integers $j$ and $t$ such that $t > \lambda(j)$,

$$\mu(j) = -\sum_{k=1}^{t} (-1)^k \binom{t}{k} d_{k-2}(j).$$

*Solution by Leonard Carlitz, Duke University.* Define $\Delta_t(m)$ by means of

$$\Delta_t(m) = \sum_{r_1 r_2 \cdots r_t = m} 1 \qquad (r_1, \cdots, r_t > 1),$$

where the order of the factors of $m$ must be taken into account, as also in the definition of $d_k(m)$. Evidently $\Delta_t(m) = 0$ for $t > \lambda(m)$. Then it is clear that

$$(1) \qquad (1 - \zeta(s))^t = \left(-\sum_{m=2}^{\infty} m^{-s}\right)^t = (-1)^t \sum_{m=2^t}^{\infty} \frac{\Delta_t(m)}{m^s},$$

$\zeta(s)$ being the Riemann zeta-function. We have also

$$(2) \qquad (1 - \zeta(s))^t = \sum_{k=0}^{t} (-1)^k \binom{t}{k} \zeta^k(s) = \sum_{k=0}^{t} (-1)^k \binom{t}{k} \sum_{m=1}^{\infty} \frac{d_{k-1}(m)}{m^s},$$

so that comparison with (1) yields

$$(3) \qquad \sum_{k=0}^{t} (-1)^k \binom{t}{k} d_{k-1}(m) = (-1)^t \Delta_t(m).$$

In particular we have

$$(3') \qquad \sum_{k=0}^{t} (-1)^k \binom{t}{k} d_{k-1}(m) = 0 \qquad (t > \lambda(m)).$$

In the same way, multiplying both sides of (2) by $(\zeta(s))^{-1}$, we obtain

$$(4) \qquad \mu(m) + \sum_{k=1}^{t} (-1)^k \binom{t}{k} d_{k-2}(m) = (-1)^t \sum_{rs=m} \mu(r) \Delta_t(s),$$

and in particular

$$\mu(m) + \sum_{k=1}^{t} (-1)^k \binom{t}{k} d_{k-2}(m) = 0 \qquad (t > \lambda(m)), \tag{4'}$$

which is the required result. While (3′) is not stated by the Proposer it is an interesting analog of (4′). It is evident how additional formulas can be obtained without difficulty.

Also solved by T. M. Apostol, and the Proposer.

*Editorial Note.* Apostol calls attention to the formula

$$\Delta_1(n) - \Delta_2(n) + \Delta_3(n) - \cdots = -\mu(n) \qquad (n > 1) \tag{5}$$

which is equivalent to the above results. See H. S. Zuckerman, On some formulas involving the divisor function, *Bulletin of the American Mathematical Society*, v. 49 (1943), pp. 292–298 for a proof of (5) and several other such formulas. Zuckerman attributes (5) to Viggo Brun.

### Circle of Best Fit

4542 [1953, 337]. *Proposed by C. D. Olds, San Jose State College, California*

The $n$ points $(x_i, y_i)$, $i = 1, 2, \cdots, n$, should lie on a circle but they fail to do so. What circle shall we take which most nearly fulfills the condition that $\sum d_i^2$ is a minimum, where

$$d_i = [(h - x_i)^2 + (k - y_i)^2]^{1/2} - r?$$

The origin of coordinates is at the centroid of the given points; and $(h, k)$ is the center, $r$ the radius, of the circle to be found. If necessary, assume that $d_i$ is small compared with $r$.

*Solution by Hermann von Schelling, Groton, Connecticut.* The difficulty of the problem stems from an artificial mathematical formulation. Requested is the circle which gives the "best" fit to $n$ points, $P_i$. Let us assume that we know this circle. We denote the distances of $P_i$ from the circle by $d_i$ and $D_i$, where $d_i$ is the numerically smaller one. The proposer starts from the condition

$$\sum d_i^2 = \text{minimum.} \tag{1}$$

It is artificial since there are two distances both of which should be considered. A more natural condition reads

$$\sum (-d_i D_i)^2 = \text{minimum.} \tag{2}$$

It is equivalent with the problem of finding three constant parameters $a$, $b$, $c$ so that

$$\sum (x_i^2 + y_i^2 - 2ax_i - 2by_i - c)^2 = \text{minimum.} \tag{3}$$

The classic method of least squares yields the solution.

If $d_i$ is small compared with $2r$, then $D_i$ becomes practically constant and (2) is reduced to condition (1). In this sense it may be said that the solution of (3) "most nearly" fulfills the condition (1) of the proposer.

Also solved by O. E. Stanaitis.

## RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

*All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.*

*Calculus.* By G. B. Thomas. Cambridge, Addison-Wesley Publishing Company, Inc. 1953. 614 pages. $6.50.

This text would be suitable for a four-semester introductory course in calculus. In addition it contains several topics usually encountered in a course in advanced calculus. The order of presentation and the manner of exposition are such as to make it fall in the category of "traditional" calculus texts.

The first three chapters are devoted to a study of derivatives and differentials, while the next two introduce the topic of integration. The indefinite integral is encountered first, but emphasis is placed on the definite integral. A short chapter on polar coordinates is followed by one on the transcendental functions with a separate chapter on the hyperbolic functions. Chapter nine contains a systematic study of the technique of integration. In the next chapter vectors are introduced and are used in the subsequent chapters on solid geometry and partial differentiation. The remaining topics in the book are multiple integrals, infinite series, and differential equations.

The book has many good features. Throughout there are numerous examples to illustrate the important features of definitions and theorems. In addition there is an ample supply of drill problems and also some of greater difficulty to challenge the better student. The diagrams in the book are good; in fact the whole format is pleasing. The use of vectors in the chapters on solid geometry and partial differentiation appealed to the reviewer, as did the chapter on infinite series.

Misprints appeared to be few, only one causing any difficulty. This was in the index where a reference to *acceleration* is given on page 19, but the word does not occur on that page.

Those who like a text to be concise might not care for the way in which everything is spelled out in great detail. But to the reviewer it seemed like a good book, and compares favorably with texts well-established in the field.

W. FRASER
Toronto

*Higher Transcendental Functions*, Vols. I and II based, in part, on notes left by Harry Bateman. By the staff of the Bateman manuscript project (Arthur Erdélyi, Director; Wilhelm Magnus, Fritz Oberhettinger, Francesco G. Tricomi). New York, McGraw-Hill Book Company, 1953. Vol. I, 26+302 pages. $6.50. Vol. II, 17+396 pages. $7.00.

A review could hardly improve on what Erdélyi himself has written in description of this work (pp. xv–xvii). "The work of which this book is the first volume might be described as an up-to-date version of *Part II. The Transcendental Functions* of Whittaker and Watson's celebrated 'Modern Analysis'. Bateman (who was a pupil of E. T. Whittaker) planned his 'Guide to the Function' on a gigantic scale. In addition to a detailed account of the properties of the most important functions, the work was to include the historic origin and definition of, the basic formulas relating to, and a bibliography for *all* special functions ever invented or investigated. These functions were to be catalogued and classified under twelve different headings according to their definition by power series, generating functions, infinite products, repeated differentiations, indefinite integrals, definite integrals, differential equations, difference equations, functional equations, trigonometric series, series of orthogonal functions, or integral equations. Tables of definite integrals representing each function and numerical tables of a few new functions were to form part of the 'Guide'. An extensive table of definite integrals and a Guide to numerical tables of special functions were planned as companion works.

"The great importance of such a work hardly needs emphasis. Bateman's unparalleled knowledge of the mathematical literature, past and present, and his equally exceptional diligence, would have made the book an authoritative account of its vast subject, and in many respects a definitive account; a Greater Oxford Dictionary of special functions.

"A realistic appraisal of our abilities and of the time at our disposal led to a drastic revision of Bateman's plans. Only Bateman himself had the erudition to give a reliable and accurate history of special functions, and the man power available to us was insufficient for the inclusion of all functions. Thus we restricted ourselves to an account (probably far less detailed than that planned by Bateman) of the principal properties of those special functions which we considered the most important ones. The loss thus caused to mathematical scholarship is great, regrettable, and final, but we venture to hope that it will be counterbalanced in some measure by the considerable reduction in size of the book, and by the gain in the clarity of its organization. We can only hope that although the scope of the present work is much narrower than that envisaged by Bateman, in its humbler sphere the book will be more useful. . . .

"For the most part we were unable to make extensive use of Bateman's voluminous notes: we found it easier to compile our account of the various functions from our knowledge of these functions supplemented by routine search in the available literature."

A few mathematicians will be disappointed by this work. To them I can remark that, having known Bateman toward the end of his life, I think it unlikely he would have finished either of his planned works no matter how long he had lived. As can be seen from the sequence of books he did finish, in him the characteristic British passion for disorder grew with age to monstrous intensity and the huge task of organization before actually beginning to write these two works he was reluctant to face. On the other hand, the staff which has written the work under review has taken up a large responsibility, for to them alone was given any opportunity of making use of the great mass of material left by Bateman, an opportunity, as they tell us, they have decided to decline. It seems unfair both to Bateman and to the distinguished authors themselves that Bateman's name, not theirs, appears on the title page, which is cluttered besides with government gobbledegook.

The analysts of the twenties and thirties turned away from the "classical" approach with its formulae and explicit calculation, preferring instead to seek generality, method, and idea. To many trained in this "modern" line of thought, works such as Bateman planned are a voice from the past, of interest only to "applied mathematicians." Perhaps there was a trace of this view behind the decision to write a handbook instead of a treatise. If, as one often hears, special functions are used only by numerical practitioners who do not happen to have a large XXXAC at their disposal, the compactness and selection of the present work are advantages. We note that on one of the back covers the publisher advertises engineering handbooks. But, while the children of the "classical" analysts tend to refer to them with a shade of patronism, sometimes grandchildren are closer than children. With the resurgence of applied *mathematics* (not merely *application* of mathematics), so noticeable a trait of the mathematical scene of the fifties, it is possible that exhaustive treatises in analysis may come into the regard that the classical exhaustive treatises on hydrodynamics and elasticity have again and rightly been granted. That the handbook scheme permits a reduction in *size*, to which the introduction several times refers, may not always be a recommendation. It is to be hoped that Bateman's notes remain intact for possible future use.

Turning now to what these books are rather than what they are not, we find them in every respect admirable examples of what a handbook should be. The authors have written a first rate Webster's Collegiate without the benefit of a Greater Oxford as source. In making this project possible the Office of Naval Research has done service to many parts of science. Most of the material is available in no other book. While it is easy to find misprints in the text and bibliography, I have not searched for errors in the formulae themselves, since I doubt if there are any. The work has been carried out in the spirit that reliability comes first, and in a project of this type and size the labor of checking must be very heavy indeed.

The organization is by functions: gamma, hypergeometric, Legendre, generalizations of the hypergeometric (several types, two chapters), confluent

hypergeometric, Bessel, parabolic and paraboloidal, incomplete gamma, orthogonal polynomials, spherical and hyperspherical, orthogonal polynomials in several variables, elliptic. Most of the chapters begin with theory, followed by a table of formulae. There is an attempt to compromise between a pure list of results and an exposition of the subject. More difficult theorems are stated without proof, but when the steps in deriving a formula are fairly easy they are indicated. References tend to be to relatively recent works, especially when there exists a standard treatise for the function in question. There is a brief index of subjects, but no index of names; in the text, the discoverer of a formula is sometimes noted, and if so, accurately, but usually without a reference. There is little if any attempt to provide the reader with methods of attack for new problems. All kinds of readers will welcome heartily the careful and precise statement of range of validity which accompanies every formula.

Unfortunately it is necessary to complain about the format. The book is planographed, not printed. While the result is readable, it is not pretty. The typewriter was a good one as typewriters go, but the typing is ragged. For example, on the dedication page the first letter is out of line, as are many of the capitals throughout, and on nearly any page one sees letters run together or stretched apart. These books will certainly remain the standard in their field, which has a wider audience than most in mathematics, for a quarter century or longer. Every serious student of analysis or of any branch of applied mathematics will wish to own them, and the publisher had all reasonable assurance of a steady moderate sale for many years. To issue these monuments in a format suggesting a government progress report seems to me not only an abuse of the copyright privilege and a virtual insult from the publisher to the authors but also a major underestimate of a commercial opportunity.

C. TRUESDELL
Indiana University

## NEW BOOKS RECEIVED

*Henri Poincaré.* By Tobias Dantzig. New York, Charles Scribner's Sons, 1954, xi+149 pages. $3.00.

*College Algebra.* By H. G. Apostle. New York, Henry Holt and Company, 1954, xiii+432 pages. $4.50.

*Intermediate Algebra.* By L. J. Adams. New York, Henry Holt and Company, 1954, xiv+370 pages. $3.40.

*Introduction to Elliptic Functions.* By F. Bowman. New York, John Wiley and Sons, Inc., 1954. 115 pages. $2.50.

*Mathematics of Statistics, Part One.* Third Edition. By J. F. Kenney and E. S. Keeping. New York, D. Van Nostrand Company, Inc., 1954. xiii+346 pages. $5.00.

*Geometrical Mechanics and De Broglie Waves.* By J. L. Synge. New York, Cambridge University Press, 1954. ix+167 pages. $4.75.

*Introduction to Mathematical Statistics*. Second Edition. By P. G. Hoel. New York, John Wiley and Sons, Inc., 1954. xi+331 pages. $5.00.

*Transactions of the Symposium on Fluid Mechanics and Computing*. Volume I. Sponsored by The American Mathematical Society and the Office of Ordnance Research, U. S. Army. New York, Interscience Publishers, Inc., 1954. 243 pages. $5.00.

*Mathematics in Agriculture*. Second Edition. By R. V. McGee. New York, Prentice-Hall, Inc., 1954. x+208 pages. $5.35.

*Analytic Geometry*. By Gordon Fuller. Cambridge, Massachusetts, 1954. ix+205 pages.

*Theory of Games and Statistical Decisions*. By David Blackwell and M. A. Girshick. New York, John Wiley and Sons, Inc., 1954. ix+355 pages. $7.50.

*Calculus*. Third Edition. By G. E. F. Sherwood and A. E. Taylor. New York, Prentice-Hall, Inc., 1954. xv+579 pages. $7.65.

*Calculus*. By G. M. Merriman. New York, Henry Holt and Company, 1954. xii+625 pages. $6.50.

*Differential Equations With Applications*. By Herman Betz and P. B. Burcham and G. M. Ewing. New York, Harper and Brothers Publishers, 1954. x+310 pages. $4.50.

*Introductory Calculus With Analytic Geometry*. By E. G. Begle. New York, Henry Holt and Company, 1954. x+304 pages. $4.50.

*Elementary Theory of Numbers*. By Harriet Griffin. New York, McGraw-Hill Book Company, 1954. ix+203 pages. $5.00.

*Differential and Integral Calculus*. Fifth Edition. By C. E. Love and E. D. Rainville. New York, The Macmillan Company, 1954. 14+526 pages. $5.75.

## NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.*

### MATHEMATICS DIVISION OF ASEE

The Mathematics Division of the American Society for Engineering Education met on June 16–18, 1954, at the University of Illinois, Urbana, Illinois. Four well-attended sessions of the Division were held. The following new officers of the Division were elected at the annual business meeting: Chairman, Professor C. O. Oakley, Haverford College; Secretary, Professor W. E. Restemeyer, University of Cincinnati; Director, Dr. R. S. Burington, Bureau of Ordnance, Navy Department. Professor H. M. Gehman, University of Buffalo, and Dr. C. V. Newsom, State University of New York, will continue to serve as Directors

and Professor H. K. Justice, University of Cincinnati, continues as Council Member of ASEE.

The next annual meeting of the ASEE Mathematics Division will be held in June 1955 at Pennsylvania State University. For further information write to Professor W. E. Restemeyer, University of Cincinnati.

### REPORT OF MANPOWER RESOURCES IN MATHEMATICS

*Manpower Resources in Mathematics*, a report on the professional characteristics, employment, and earnings of mathematicians in the United States, has been issued by the National Science Foundation. The report, prepared jointly by the Foundation and the Bureau of Labor Statistics of the United States Department of Labor, is based on information supplied to the National Scientific Register in 1951 by about 2,400 mathematicians.

The report can be obtained from the Superintendent of Documents, United States Government Printing Office, Washington 25, D. C., at a cost of twenty cents.

### RESEARCH FELLOWSHIPS IN PSYCHOMETRICS

The Educational Testing Service is offering for 1955–56 its eighth series of research fellowships in psychometrics leading to the Ph.D. degree at Princeton University. Open to men who are acceptable to the Graduate School of the University, the two fellowships each carry a stipend of $2,500 a year and are normally renewable. Fellows will be engaged in part-time research in the general area of psychological measurement at the offices of the Educational Testing Service and will, in addition, carry a normal program of studies in the Graduate School.

Suitable undergraduate preparation may consist either of a major in psychology with supporting work in mathematics, or a major in mathematics together with some work in psychology. However, in choosing fellows, primary emphasis is given to superior scholastic attainment and demonstrated research ability rather than to specific course preparation.

The closing date for completing applications is January 13, 1955. Information and application blanks will be available about November 1 and may be obtained from: Director of Psychometric Fellowship Program, Educational Testing Service, 20 Nassau Street, Princeton, New Jersey.

### CONFERENCE ON THE ROLE OF WOMEN'S COLLEGES IN THE PHYSICAL SCIENCES

A conference on the role of the women's colleges in the physical sciences was held on the Bryn Mawr College campus on June 17–18, 1954. It was attended by professors of chemistry, mathematics, and physics from seventeen independent women's colleges, by twelve research and development officials from large industries, and by representatives of government agencies, hospitals, and educational associations.

A detailed report of the conference is being printed. It may be obtained on

request from the Chairman, Dr. W. C. Michels, Bryn Mawr College, Bryn Mawr, Pennsylvania.

## PERSONAL ITEMS

Dr. E. O. Lovett, first president of Rice Institute, has been awarded the honorary degree of LL.D. by Princeton University, where he was formerly Professor of Mathematics.

Cornell University announces the following: Dr. Ilse Novak Gal and Dr. I. S. Gal have been promoted to assistant professorships; Professor H. L. Hamburger of the University of Cologne, Germany, has been appointed Visiting Professor of Mathematics for the year 1954–55; Miss Evelyn M. Bender, Mr. H. K. Flesch, Mr. H. F. G. Gardner, and Mr. M. E. Muller have been appointed to instructorships; Mr. G. E. Collins and Mr. Elliott Mendelson have been awarded National Science Fellowships; Professor R. J. Walker is on leave of absence; Professor W. A. Hurwitz has retired with the title of Professor Emeritus.

Georgia Institute of Technology announces: Assistant Professors W. B. Evans, J. H. Wahab, and R. A. Willoughby have been promoted to associate professorships; Dr. R. H. Kasriel, formerly aeronautical research scientist for the National Advisory Commission for Aeronautics, and Dr. W. M. Perel, previously a teaching assistant at Indiana University, have been appointed to assistant professorships; Professor D. M. Smith has retired.

At Massachusetts Institute of Technology: Mr. D. G. Aronson, Mr. F. G. Brauer, Mr. N. J. Hicks, Mr. J. E. Kimber, Jr., and Mr. Gustave Solomon have been appointed to instructorships.

Michigan State College makes the following announcements: Professor R. D. James of the University of British Columbia has been appointed Visiting Professor for the year 1954–55; Dr. Edward Silverman of Sandia Corporation has been appointed to an assistant professorship; Dr. J. H. McKay of the University of Washington and Dr. Andre Laurent, a postdoctoral fellow at the University of Chicago, have been appointed to instructorships; Assistant Professor Henry Parkus has been promoted to an associate professorship; Dr. J. G. Hocking and Dr. J. B. Kelly have been promoted to assistant professorships; Assistant Professor Mary H. Payne has resigned; Dr. H. A. Hanson has resigned to accept an associate professorship at Upsala College; Dr. C. H. Kraft has resigned to accept an assistant professorship at the University of California.

State Teachers College at Montclair, New Jersey, announces: Professor D. R. Davis has been appointed Chairman of the Department of Mathematics, replacing Professor V. S. Mallory who has retired; Assistant Professor B. E. Meserve of the University of Illinois has been appointed to an associate professorship.

University of Colorado reports that Professor B. W. Jones of the University and Professor K. A. Hirsch of Queen Mary College of London have exchanged positions for the academic year of 1954–55.

University of North Carolina announces that a symposium on multivariate statistical analysis was held at the University on April 21–24, 1954. About 35 statisticians, psychologists and economists attended. The formally announced participants were M. G. Kendall, C. R. Rao, Harold Hotelling, S. N. Roy, R. C. Bose, R. L. Anderson, Aleyamma George, Seymour Geisser and Earl Diamond, but numerous others contributed also. Considerable unpublished material and many new ideas were brought to light in fields such as factor analysis, multicollinearity, discriminant functions and related classification problems, basic criteria for multipleparameter estimation, canonical correlation, serial correlation, $T$ statistics, and multivariate confidence bounds.

University of South Carolina announces the following: Professor Tomlinson Fort of the University of Georgia has been appointed to a professorship; Dr. Eckford Cohen, previously at the Institute for Advanced Study, and Dr. D. D. Strebe of the University of Buffalo have been appointed to assistant professorships.

Mr. R. A. Barnett, previously a teaching assistant at the University of Southern California, is teaching now in the Oakland Public Schools, California.

Dr. Archie Blake, formerly head of the I.B.M. Computing Section, Cornell Aeronautical Laboratory, Buffalo, New York, is now an advisory engineer for the Westinghouse Electric Corporation, Baltimore, Maryland.

Associate Professor R. C. Boles of State Teachers College, Florence, Alabama, has been appointed to an associate professorship at Mercer University.

Mr. E. N. Brandt, Jr., previously a student at the University of Oklahoma, has been appointed to a graduate assistantship at Oklahoma Agricultural and Mechanical College.

Assistant Professor J. R. Byrne of San Jose State College has been appointed to an instructorship at Portland State Extension Center, Oregon.

Mr. T. E. Cheatham, Jr. and Mr. J. W. Smith, graduate students at Purdue University and participants in a fellowship program established by Electro-Data Corporation, were given a ten-week computer instruction course at ElectroData's computing center, Pasadena, California.

Mr. D. R. Childs, recently a student at the University of New Hampshire, has been appointed to a graduate assistantship at Vanderbilt University.

Mr. K. J. Cohen, former fellow at Cornell University, is now National Science Foundation Predoctoral Fellow at Carnegie Institute of Technology.

Assistant Professor K. L. Cooke of State College of Washington is on leave of absence as a research associate at Massachusetts Institute of Technology for the year 1954–55.

Mr. E. L. Dubowsky, previously a graduate assistant at Kansas State College, is teaching now in Colby Community High School, Kansas.

Professor Howard Eves of Harpur College has accepted a position as Visiting Professor for the academic year 1954–55 at the University of Maine.

Associate Professor E. L. Godfrey of Defiance College has a position as a mathematician at Wright-Patterson Air Force Base, Dayton, Ohio.

Mr. G. R. Grainger, recently a graduate student at the University of Notre Dame, is employed now as a research engineer by Consolidated Aircraft Corporation, San Diego, California.

Mr. J. E. Guinane, previously a graduate student at California Institute of Technology, is now with the Raytheon Manufacturing Company, Waltham, Massachusetts.

Professor T. R. Hollcroft, chairman of the Department of Mathematics of Wells College, has retired with the title of Professor Emeritus; he has been appointed Historian of the College.

Assistant Professor S. L. Jamison of Florida State University is now Applied Science Representative for International Business Machines, Los Angeles, California.

Mr. A. T. Kovitz, former instructor at Polytechnic Institute of Brooklyn, is now with the Arma Corporation, Roosevelt Field, New York.

Mr. Rolando Lara, formerly with the Seismograph Service Corporation, Tulsa, Oklahoma, is employed now by the Lorac Service Corporation, Morgan City, Louisiana.

Mr. Elmer Latshaw, recently a technical engineer with ACF Brill Motors Company, Philadelphia, Pennsylvania, has a position as a mechanical engineer at Naval Air Material Center, Philadelphia.

Dr. C. E. Lemke of Carnegie Institute of Technology has accepted a position as a research associate at the Knolls Atomic Power Laboratory, General Electric Company, Schenectady, New York.

Dr. M. D. Marcus, who has been a research assistant at the University of California, has been appointed to an instructorship at the University of British Columbia.

Dr. A. D. Martin of Oberlin College is now at the Institute for Advanced Study.

Mr. R. D. Mayer, recently a student at Purdue University, has been appointed to a teaching assistantship at the University of Washington.

Mr. R. J. Mercer has accepted a position at the Naval Electronics Laboratory, San Diego, California.

Mr. R. B. Merrill, formerly of the Bureau of Ships, Washington, D. C., is now a sales correspondent with the Eagle-Picher Company, Chicago, Illinois.

Assistant Professor D. G. Miller of the University of Louisville is now Research Associate, Department of Chemistry, Brookhaven Laboratory, Upton, New York.

Mr. Leonard Miller, who has been associated with the New York Bottle & Closure Company, New York City, has accepted a position with the Abbey Northern Glass Corporation, Brooklyn, New York.

Professor C. N. Mills has retired from his position as Head of the Department of Mathematics of Illinois State Normal University and has joined the faculty of Augustana College.

Miss Mardell S. Miskowski, formerly a graduate assistant at Marquette

University, has a position as a mathematician at Aberdeen Proving Ground.

Mr. Robert E. Montgomery of Trinity College, Hartford, Connecticut, is now at the United States Naval Research Laboratory, Washington, D. C.

Dr. F. R. Olson of Duke University has been appointed to an assistant professorship at the University of Buffalo.

Dr. A. M. Peiser, who has been engaged in private consulting in Wantagh, New York, is now Head of the Electronic Computer Section, M. W. Kellogg Company, Jersey City, New Jersey.

Mr. D. W. Pounder has been transferred from deHavilland Propellers Limited, England, to deHavilland Aircraft of Canada Limited.

Professor Edward Rayher of Bergen Junior College, Teaneck, New Jersey, has been appointed to an instructorship at Fairleigh Dickinson College.

Mr. F. A. Raymond, recently a student at Harvard University, has been appointed to a teaching fellowship at the University of Michigan.

Dr. Charles Roth, formerly guidance counselor at City College of the City of New York, is engaged now as a consulting psychologist by John R. Martin Associates, New York City.

Associate Professor A. J. Sachs has been promoted to the position of Professor of the History of Mathematics at Brown University.

Mr. L. A. Schmidt has been promoted to the position of Supervisor, Aerodynamics and Compressible Flow Section, Armour Research Foundation of Illinois Institute of Technology.

Dr. Wladimir Seidel, who has been at the Institute for Advanced Study, has been appointed to a professorship at the University of Notre Dame.

Assistant Professor E. L. Stanley of Clemson Agricultural College has been promoted to an associate professorship.

Mr. C. E. Stewart, previously a research assistant at the Armour Research Foundation of Illinois Institute of Technology, has accepted a position at the Boeing Airplane Company, Seattle, Washington.

Mr. C. R. Strain, who has been a teaching fellow at the University of Michigan, is employed now as a mathematician by Engineering Research Associates, Arlington, Virginia.

Dr. J. L. Ullman of the University of Michigan has been promoted to an assistant professorship.

Assistant Professor R. Z. Vause, Jr., of Clemson College has been appointed to an assistant professorship at the University of North Carolina.

Professor V. H. Wells of Williams College has retired.

Mr. Nathan Wetrogan, formerly with the United States Navy Hydrographic Office, Washington, D. C., has a position at the Lockheed Aircraft Corporation, Marietta, Georgia.

Mr. L. H. Williams, recently a graduate assistant at the University of Georgia, is now on active duty in the United States Navy and is at the United States Naval Proving Ground, Dahlgren, Virginia.

Dr. W. E. Wilson of the Office of Ordnance Research, Durham, North

Carolina, has been appointed to a professorship in the College of Engineering, Pennsylvania State University.

Mr. F. W. Darling, who had retired from his position with the United States Coast and Geodetic Survey, Washington, D. C., died on April 21, 1954. He had been a member of the Association for thirty-four years.

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## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 81 persons have been elected to membership by the Board of Governors on applications duly received.

G. L. ALEXANDERSON, Student, University of Oregon.

MARY N. APPLEGATE, Student, University of Oklahoma.

D. D. AUFENKAMP, Doctorat d'Universite (Paris) Instr., Reed College.

C. W. BARNETT, M.S. (Louisiana S. U.) Jr. Mathematician, Lockheed Aircraft Co., Marietta, Ga.

J. C. BARTLETT, Student, Kentucky Wesleyan College.

J. A. BEANE, M.A. (Buffalo) Professor and Head of Department of Drawing, Mechanics and Design, School of Engineering, University of Buffalo.

D. C. BENSON, M.S. (Iowa S. C.) Instr., Iowa State College.

W. H. BEYER, M.S. (V.P.I.) Mathematician, Goodyear Aircraft Corp., Akron, Ohio.

KURT BING, Ph.D. (Harvard) Asst. Professor, Rensselaer Polytechnic Institute.

NORBERT BISCHOF, Student, Syracuse University.

J. H. BLAU, Ph.D. (North Carolina) Asso. Professor, Antioch College.

A. A. BLUNDI, B.A. (LaSalle) Computing Systems Engineer, Burroughs Corp.

C. E. BUIE, M.A. (U.C.L.A.) Teacher, Pasadena City College.

ARTHUR BURGER, Student, New Jersey State Teachers College, Upper Montclair.

MARIA CASTELLANI, Ph.D. (Rome) Asso. Professor, University of Kansas City.

D. A. COPE, Engineering Asst., Glenn L. Martin Co., Baltimore, Maryland.

JOSEPH DEMILIA, B.S. in E.E. (C.C.N.Y.) Electrical Engineer, U. S. Naval Gun Factory.

J. E. DERWENT, Student, University of Notre Dame.

DIANA DODGE, B.A. (Smith) Grosse Pointe Farms, Michigan.

A. H. EVANCIC, M.A. (Pittsburgh) University of Pittsburgh.

C. H. FINNIE, JR., B.S. (Southwest Texas S.) Lt. USAF, Student, University of Calif.

ANNE J. FLANAGAN, B.A. (Seton Hill) Assistant, Florida State University.

H. J. FLETCHER, Ph.D. (Utah) Instr., Brigham Young University.

R. M. GARDNER, Student, Arizona State College at Flagstaff.

R. L. GAY, M.S. (North Carolina S. C.) Asst. Professor, Wake Forest College.

JOHN GURLAND, Ph.D. (California) Asso. Professor, Iowa State College.

J. R. HAMILTON, M.S. (N.Y.U.) Asst. Professor, Long Island University.

J. H. HAMMER, Studienreferendar, Oberschule, Bremen, Germany.

W. J. HARROLD, Student, Polytechnic Institute of Brooklyn.

R. M. HENDRICKS, B.A. (Santa Barbara C.) Lt.(j.g.), United States Navy.

J. T. HINELY, JR., Student, University of Georgia.

G. C. HOLT, M.A. (Tennessee) Aerophysics Engineer, Consolidated-Vultee Aircraft Co., Fort Worth, Texas.

J. E. HOULE, JR., M.A. (Catholic) Instr., Georgetown University.

MRS. MARY R. HUDSON, M.A. (Peabody) Asso. Professor, Florence State Teachers College.

H. L. HUNZEKER, M.S. (Iowa S. C.) Instr., DePauw University.

KENJI INOUYE, M.A. (Columbia) Teacher, Hilo High School, Hawaii.

WALTER JAMES, B.S.Ch.E. (Minnesota) Instr., University of Minnesota.

S. R. JONES, Student, Harvard University; Design Engr., Barkley & Dexter Labs., Boston, Mass.

R. E. KELLY, B.S. (Duquesne) Mathematician, U. S. Bureau of Mines, Pittsburgh, Pa.

MAURICE KENNEDY, M.S. (Ireland) Grad. Student, California Institute of Technology.

N. K. KIM, Student, University of Oregon.

O. M. KLOSE, S.M. (Chicago) Asst. Professor and Head of Mathematics Department, Seattle University.

L. A. KOKORIS, Ph.D. (Chicago) Asst. Prof., University of Washington.

M. L. KROM, B.A. (Iowa) Radio Repairman, United States Army.

AARON LIEBERMAN, M.S. (N.Y.U.) Mathematician, Republic Aviation Corp., Farmingdale, N. Y.

J. P. LIPP, Student, University of Oklahoma.

L. H. LOOMIS, Ph.D. (Harvard) Asso. Professor, Harvard University.

ELIHU LUBKIN, Student, Columbia University.

L. C. MARSHALL, M.A. (Pennsylvania) Mathematician, White Sands Proving Ground, New Mexico.

PEREA MCCANE, B.A. (Seton Hill) Mathematician, Aberdeen Proving Ground, Maryland.

R. V. MCGEE, M.S. (Texas A.&M.) Asso. Professor, Agricultural and Mechanical College of Texas.

D. T. MITCHELL, Student, Wabash College.

C. B. MOORE, Ph.D. (Kentucky) Senior Aerophysics Engr., Consolidated-Vultee Aircraft Corp., Fort Worth, Texas.

BARBARA A. O'CONNELL, Student, Regis College.

L. N. ORLOFF, M.A. (California) Draftsman, Pacific Gas and Electric Co., San Francisco, California.

G. W. PEGLAR, Ph.D. (Iowa) Asst. Professor, Iowa State College.

J. H. PIERSON, B.S. (New Mexico) Albuquerque, New Mexico.

R. L. PLUNKETT, Ph.D. (Virginia) Asst. Professor, Florida State University.

L. B. PORTER, M.S. (Michigan) Actuarial Asst., North Carolina Mutual Life Insurance Co., Durham.

K. E. RALSTON, Student, Massachusetts Institute of Technology.

JOHN RAUSEN, M.A. (Columbia) Instr., University of Connecticut.

BERNARD REIMANN, Studienrat, Wilhelm-Gymnasium, Hamburg, Germany.

P. M. RINEARSON, B.S. (Washington) Research Asst., University of Washington.

L. C. ROBBINS, JR., M.A. (Pennsylvania) Supervisor of Programming, Burroughs Corp., Pa.

C. A. ROGERS, M.S. (Washington) Colorado Agricultural and Mechanical College, Fort Collins, Colorado.

MARILYN A. ROGERS, Student, Carleton College.

W. H. SAWYER, Student, University of Oklahoma.

S. A. SIMS, M.S. (Texas A.&M.) Asst. Professor, Agricultural and Mechanical College of Texas.

SISTER MARION BEITER, M.S. (St. Bonaventure) Instr., Rosary Hill College.

L. M. SOULE, B.A. (Alfred) Jr. Instr., Agricultural and Technical Institute, Alfred, N.Y.

J. C. SPRADLING, B.S. (Missouri) Teacher, Wisconsin Institute of Technology.

R. W. STEWART, Student, Arizona State College at Tempe.

M. L. TOMBER, Ph.D. (Pennsylvania) Instr., Amherst College.

E. A. TRAYNOR, Student, Holy Cross College.

FRANCES L. WALKER, B.A. (Oklahoma C. for Women) Grad. Asst., University of Oklahoma.

J. N. WALLACE, M.A. (Duke) Long Branch, New Jersey.

C. B. H. WATSON, Student, University of Toronto.

H. J. WEISS, D.Sc. (Carnegie) Asst. Professor, Iowa State College.

B. G. WILLIS, Student, Oklahoma Agricultural and Mechanical College.

C. E. YINGST, Student, Lebanon Valley College.

P. W. ZEHNA, A.M. (Colorado S. C.) Acting Instr., Colorado State College of Education.

## THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The twenty-eighth meeting of the Allegheny Mountain Section of the Mathematical Association of America was held on May 1, 1954, at Marshall College, Huntington, West Virginia. Professor Frederick H. Steen, Chairman of the Section, presided at the morning and afternoon sessions.

There were twenty-eight in attendance including the following twenty-one members of the Association:

A. N. Aheart, A. G. Anderson, H. W. Baeumler, J. J. Barron, E. J. Cogan, A. B. Cunningham, H. A. Davis, A. R. Erskine, Mary A. Goins, Hunter Hardman, E. E. High, J. C. Knipp, L. T. Moston, B. H. Mount, Jr., Pauline E. Mount, E. F. Myers, Ruth E. O'Donnell, L. A. Ondis II, Morris Ostrofsky, F. H. Steen, J. K. Stewart.

Dr. Frank Bartlett, Dean of the College of Arts and Sciences, welcomed the people to Marshall College. Then the following papers were presented.

1. *Operation of the analytical group in industry*, by Dr. B. H. Mount, Jr., Westinghouse Electric Corporation.

The author described the personnel and the computing devices used to solve the engineering problems in a certain industrial analytical group. Certain mechanical and electrical problems were described and their solutions on analog and digital computers were outlined.

2. *An asymptotic expansion of a certain infinite integral*, by Professor Nelson Yeardley, Thiel College, introduced by the Chairman.

An asymptotic expansion of the integral

$$\int_0^\infty \exp\left(-x[v(1-2^\alpha/x)/4+4q/vx]\right)v^{-3/2}(1-e^{-v})^{-1/2}dv, \qquad x>0, \alpha>-1, v \text{ complex},$$

is obtained by using an adaption of Van der Waerden's method as described in his paper, *On the method of saddle points*, Applied Scientific Research, v. B2, no. 1B, 1951. The fact that the exponent is a function of $x$ as well as $v$ produces complications which do not occur in Van der Waerden's problem. These complications are resolved.

3. *Survey courses in mathematics for the liberal arts student*, by Mr. E. J. Cogan, Pennsylvania State University.

The purposes of survey courses in mathematics were discussed. Among them are: to provide closure in communication; to explain the terms: truth, validity, proof; to introduce the ability to construct mathematical models for various systems; to furnish a background of basic ideas to the student of mathematics and science. Special techniques of presenting material were suggested as well as certain lists of topics. Among the course ideas were: basic analysis; the reading of data numerically presented; logic and foundations; mathematical games. A list of available textbook material was given.

4. *Hypercomplex systems in genetics*, by Professor A. G. Anderson, Duquesne University.

The manner in which various linear non-associative algebras lend themselves to problems involving diverse types of inheritance was discussed, with particular emphasis upon a very simple hypercomplex system which finds applications throughout the field of genetics. The commutative algebra having the multiplication table $XX=X$, $YY=X/4+Y/2+Z/4$, $ZZ=Z$, $XY=X/2+Y/2$, $XZ=Y$, $YZ=Y/2+Z/2$ is the zygotic algebra of simple Mendelian inheritance, but also finds use in connection with the prediction of quantitative characteristics in polygenic systems. Genetic algebras for asymmetrical inheritance also exist.

5. *Scoring problems in algebra*, by Dr. G. R. Lewis, Clarion State Teachers College, introduced by the Chairman.

An algebra test consisting of five problems of varying difficulty was administered to 101 students in five high schools. The tests were scored on a right-wrong basis. Then, each problem was analyzed and credit assigned to the steps in the solution, scores being obtained on this basis. Coefficients of correlation were computed and differences in mean scores determined and tested for significance. It was discovered that the differences in means were non-significant. The coefficients of correlation were high enough to warrant concluding that a student's rank was predicted by a raw score as adequately as by a computed score.

L. T. MOSTON, *Secretary*

## THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-third annual meeting of the Illinois Section of the Mathematical Association of America was held at Knox College, Galesburg, Illinois, on Friday afternoon and Saturday forenoon, May 14 and 15, 1954. Professor M. C. Hartley, Chairman of the Section, presided at all sessions.

There were forty-three in attendance, including the following thirty-two members:

Beulah M. Armstrong, H. G. Ayre, J. W. Beach, Max Beberman, Imogene C. Beckemeyer, H. R. Beveridge, A. H. Black, A. O. Boatman, J. R. Brown, John Christopher, M. D. Eulenberg, A. E. Gault, J. S. Georges, A. E. Hallerberg, M. C. Hartley, E. C. Kiefer, Evelyn K. Kinney, A. O. Lindstrum, Jr., W. C. McDaniel, A. W. McGaughey, B. E. Meserve, E. B. Miller, W. L. Miser, G. E. Moore, M. G. Moore, Gordon Pall, L. L. Pennisi, Annette Sinclair, S. Grace Smyth, Rothwell Stephens, C. J. Stowell, and Arnold Wendt.

At the business meeting on Saturday morning the following officers were elected for the coming year: Chairman, Professor Rothwell Stephens, Knox College; Vice-Chairman, Professor Hugh Beveridge, Monmouth College; Secretary-Treasurer, Professor A. W. McGaughey, Bradley University. The Secretary-Treasurer's report was given and approved. The Section Governor, Professor E. B. Miller, gave a short report concerning the business of the Board of Governors during the past year. Business meeting adjourned.

The following program was presented:

1. *An integrated high school mathematics program*, by Professor Max Beberman, University of Illinois High School.

Since it is almost impossible at the ninth grade level to separate college-potential students

from terminal students, there is need for a course which can serve as common ground for all students who are ready for high school mathematics. The integrated course satisfies this requirement because it takes concepts from many areas of mathematics and of general educational value and places them in the early years of high school. More specialized needs can be met in the later years. The first course at University High includes equations, inequalities, loci, inductive and deductive reasoning, computations with measurements, trigonometry, and a set theoretic development of the cardinal numbers.

2. *Progress report on the ICP mathematics examination*, by Professor H. Glenn Ayre, Western Illinois State College.

A state-wide committee in Illinois working on college admission requirements set as one of five criteria to be used by a college or university to provide the best prediction for probable college success the following: "Score on a simple mathematical test." The Illinois Council of Teachers of Mathematics assumed the responsibility for constructing such a test.

The first task of the committee was to determine the nature and scope of the test. It was agreed to attempt to build a test that would assist in determining a student's ability to do college work and measure competency in simple mathematics, but it would not attempt to measure achievement in high school mathematics nor indicate a student's preparation for college mathematics courses. The content should, for the most part, cover arithmetic and general mathematics with emphasis on basic concepts, principles, and applications.

The test items are now in the hands of the Associate Director of the Illinois Curriculum Program who has assumed responsibility for providing the professional work necessary to put the test in final form and establish norms.

3. *The small college and the new engineering requirements*, by Professor M. Anice Seybold, North Central College.

The University of Illinois School of Engineering has new entrance requirements which amount essentially to four years of high school mathematics. The speaker suggests a freshman course in analytic geometry and the calculus combined for students who satisfy these requirements. These students would probably need some review of high school mathematics. Textbooks are already available for the combined course and have been in use in the eastern part of the United States for some time. Most colleges would want their students to have about two or three more hours of calculus. Such a course in Intermediate Calculus would be relatively easy for the small college to arrange. At the end of the freshman year, students would have the necessary calculus prerequisite for most junior and senior courses now requiring calculus.

The speaker closed with an appreciation of the benefits mathematics departments will reap from the new engineering requirements and a warning that performance of students in new mathematics courses, such as combined analytic geometry and calculus, should be checked carefully and often against the performance of students in traditional courses.

4. *Report of the committee on the strengthening of teaching mathematics in secondary schools*, by Professor B. E. Meserve, University of Illinois.

This year the Committee has been sponsored by both the Illinois Section of the Mathematical Association of America and the Illinois Council of Teachers of Mathematics. Most of the Committee's activities may be considered as implementations of its report last year (See this MONTHLY, vol. 60: pp. 652–661). It has been primarily concerned with (1) strengthening the State Department of Education's forthcoming "Guide to Supervision, Evaluation, and Recognition of Illinois Schools (Kindergarten through Junior College)" and (2) sponsoring the contest prepared by the Metropolitan New York Section of the Mathematical Association of America. Fifty Illinois high schools and over two thousand students are participating in the contest this year.

5. *Straight lines and curved lines*, by Professor W. M. Johnson, Department of Architecture, University of Illinois. (By invitation.)

A non-technical presentation offering a few suggestions on rhythm, balance, harmony and color as they apply to photography. In making the presentation the speaker reviewed the basic rules of composition mentioned above and illustrated them by using straight and curved lines in a chalk-talk which was followed by a showing of 2×2 Kodachrome slides.

6. *A generalization of a theorem in schlicht function theory*, by Professor Arnold Wendt, Western Illinois State College.

Generalize the hypotheses of Lemma III, which is a special case of the uniformization principle, on page 15 of the 1950 Colloquium Publication, *Coefficient Regions for Schlicht Functions*, so that the closed arcs $I_\nu'$, $I_\nu''$ of $|z|=1$ are identified by $g_\nu(\theta)$ such that when $e^{i\theta}$ lies in $I_\nu'$, $e^{ig_\nu(\theta)}$ lies in $I_\nu''$. Here $g_\nu(\theta)$ is a real valued, continuous, monotonic decreasing function of the real variable $\theta$, $g_\nu'(\theta)$ exists except perhaps for finitely many $\theta$, and $-M<g_\nu'(\theta)<-N, 0<N<M<\infty$. It can be shown that there exists a function $w=F(z)$ having the properties of the function in the lemma referred to except that it may be only continuous on the identified arcs.

7. *The role of amateurs in number theory*, by Professor Gordon Pall, Illinois Institute of Technology.

Some past contributions by amateurs and examples of problems which have interested amateurs were given. To illustrate the need for proof, examples were given of properties of numbers which hold for small numbers, and whose first exception occurs indefinitely far out. Characteristics of amateurs were discussed.

8. *A method for representing the sum of certain series*, by Professor L. L. Pennisi, University of Illinois, Chicago Undergraduate Division.

The main result of this paper is contained in the following:

THEOREM. *If* $z\neq 0$ *and* $R\beta>-1$, *then*

$$1)\qquad \sum_{n=0}^{\infty}\frac{z^{n+1}}{(1+\beta)\cdots(n+\beta+1)}=\frac{e^z}{z^\beta}\int_0^z e^{-w}w^\beta dw,$$

*and if* $0<|z|<1$ *and* $R\beta>-1$, *then*

$$2)\qquad \sum_{n=0}^{\infty}\frac{n!z^{n+1}}{(1+\beta)\cdots(n+\beta+1)}=\frac{1}{z^\beta}\int_0^z\frac{w^\beta dw}{(1-z)+w},$$

*where* $R\beta$ *denotes the real part of the complex number* $\beta$; $w^\beta$ *denotes* $\exp[\beta L(w)]$, $L(w)$ *being an arbitrary branch of* $\log w$ *on the segment* $[0, z]$; $z^\beta$ *denotes* $\exp[\beta L(z)]$, *and the integrand is taken along the line segment joining* 0 *to* $z$.

In particular, when $z\to 0$ and $\beta$ real in (1) and (2), one obtains the sums of many interesting series.

9. *Parabolic functions*, by Professor J. S. Georges, Wright Junior College, Chicago.

Just as the circular functions are associated with the unit circle and the hyperbolic functions with the rectangular hyperbola, even so the parabolic functions may be associated with the unit parabola. The area parameter $t$ is defined to be twice the area of the parabolic sector, namely, $t=4\int_1^x\sqrt{1+x}dx-2x\sqrt{1+x}=\operatorname{cosp}^{-1}x=\operatorname{sinp}^{-1}y/2$.

The properties of the parabolic functions and of the inverse parabolic functions are discussed

in terms of continuity, differentiability, and integrability. Relations between circular, hyperbolic, and parabolic functions are pointed out in terms of the particular parameters involved.

A. WAYNE MCGAUGHEY, *Secretary*

## THE MAY MEETING OF THE INDIANA SECTION

The thirty-first annual meeting of the Indiana Section of the Mathematical Association of America was held at Rose Polytechnic Institute, Terre Haute, Indiana, on May 8, 1954. Two sessions were held at which Professor C. P. Sousley of Rose Polytechnic Institute, Chairman of the Section, presided.

There were fifty-one in attendance including the following thirty-six members of the Association:

H. W. Alexander, W. C. Arnold, A. P. Boblett, Stanley Bolks, G. E. Carscallen, W. W. Chambers, K. W. Crain, D. E. Deal, M. W. DeJonge, W. E. Edington, P. D. Edwards, R. E. Ekstrom, H. E. H. Greenleaf, J. R. Hadley, J. W. Hamblen, Ralph Hull, E. L. Klinger, L. H. Lange, G. T. Miller, Vera T. Morris, P. A. Nurnberger, Gloria Olive, T. P. Palmer, J. C. Polley, Tibor Rado, R. M. Ross, A. R. Schmidt, M. E. Shanks, W. O. Shriner, Aubrey H. Smith, C. P. Sousley, Anna K. Suter, R. O. Virts, M. S. Webster, K. P. Williams, G. N. Wollan.

The following officers were elected: Chairman, Professor H. W. Alexander, Earlham College; Vice-Chairman, Mr. R. O. Virts, Central High School, Fort Wayne, Indiana.

Professor P. D. Edwards of Ball State Teachers College reported that the Committee on Awards had awarded Association medals to five high school seniors on the basis of excellence in mathematics demonstrated in the Indiana Science Talent Search competition.

Professor Edwards also reported on the results of a study of mathematical preparation for college in Indiana, Michigan, and Illinois in which he, representing the Indiana Section, had collaborated with Professor P. S. Jones of the University of Michigan and Professor B. E. Meserve of the University of Illinois. A report on the study had been published under the heading, "Mathematical Preparation for College," in *The Mathematics Teacher*, Vol. XLV, May, 1952.

In accordance with a recent suggestion of the Board of Governors that the Section Governor be made an officer of the Section, the Constitution was amended so as to include the Section Governor as a member of the Executive Committee.

Professor L. H. Lange reported on the mathematics competition for undergraduate students in Indiana schools which has been sponsored in recent years by Valparaiso University. Following the discussion, and in part on his suggestion, a motion was passed instructing the chairman to appoint a committee to study the value of, need for, and interest in such a competition, and consider the desirability that the Section assume its sponsorship.

The following papers were presented:

1. *Global structure of the family of integral curves of differential equations*, by

Professor M. E. Shanks, Purdue University.

The speaker discussed, in an intuitive way, the problem of topologizing the integral curves of the system $\dot{x}=f(x, y)$, $\dot{y}=g(x, y)$. When the domain of the functions $f$ and $g$ is a closed rectangle containing no critical point of the system, then the natural topology on the integral curves gives a dendrite. Some special curves were considered and the fact noted that the presence of limit cycles renders such topologization impossible. Mention was made of the recent work of L. Markus, in which the notion of separatrix is fundamental, and its relation to the above problem.

2. *A note on the triangular inequality*, by Professor Gloria Olive, Anderson College.

The triangular inequality was discussed from the standpoint of the various mathematical topics it can motivate in an undergraduate seminar on this subject. The inequality was taken from one-dimensional real space into Hilbert space; on the way, a brief geometrically motivated proof, readily extended to $n$-dimensional space, was presented.

3. *An algebraic proof of the central limit theorem*, by Professor H. W. Alexander, Earlham College.

The usual proof of the central limit theorem is based on the use of moment generating functions, a device which properly belongs in graduate mathematics. The present proof makes use of the multinomial theorem, and in this connection introduces the idea of *similar terms*, that is, terms which have the same exponents in a different order. Expressions are obtained for the higher moments of the quantity $Y_n=n^{-1/2}(X_1+X_2+\cdots+X_n)$, where $X_1, X_2, \cdots, X_n$ are from a common population with zero mean. The moments are shown to approach those of the normal distribution as $n\to\infty$.

4. *Continuity and discontinuity in analysis and geometry*, by Professor Tibor Rado, Ohio State University. (By invitation.)

Professor Rado discussed a series of examples of discontinuous functions and functionals, selected from various areas in analysis and geometry, which may be used in the classroom to throw more light upon the concept of continuity itself.

5. *An application of geometric series with two ratios*, by Professor L. H. Lange, Valparaiso University.

A geometric series with two ratios converges under certain conditions and the problem of finding these conditions and the resulting sum was solved by the author subsequent to its statement by F. Watkins as E 981 in this MONTHLY. He had found, and here discussed, an application of this type of series to a problem of obscure origin: that about the bird which flies back and forth between two cars which are following a crash course. Even when generalized this problem has a trivial solution and he is searching for an application to a problem not admitting the trivial solution.

6. *Nomography from the similar triangle viewpoint*, by Professor T. P. Palmer, Rose Polytechnic Institute.

The determinantal method of proof usually employed to establish the validity of nomographic techniques obscures the simplicity of the basic ideas in the usual alignment charts. This paper develops the parallel-line type, the concurrent-line type, and the *N* type, employing nothing more advanced than the properties of similar triangles. It also develops the two-parallel-line-and-one-curve type, in this case with the aid of elementary analytic geometry. The improved simplicity should render nomographic methods accessible to many who have previously avoided them.

J. C. POLLEY, *Secretary*

## THE MAY MEETING OF THE KENTUCKY SECTION

The May meeting of the Kentucky Section of the Mathematical Assocation of America was held at the University of Kentucky, Lexington, Kentucky, on May 8, 1954. Professor Sallie Pence, Chairman of the Section, presided at the meeting.

Sixty persons attended the meeting, including the following thirty-four members of the association:

M. C. Brown, H. W. Burnette, E. A. Cameron, Esther A. Compton, J. B. Cornelison, V. F. Cowling, H. H. Downing, J. C. Eaves, R. I. Fields, Clarence Ford, A. W. Goodman, Beulah Graham, Charles Hatfield, Aughtum S. Howard, Tadeusz Leser, A. G. McGlasson, W. L. Moore, R. S. Park, Mary Pettus, D. W. Pugsley, Sara L. Ripy, G. G. Roberts, W. J. Robinson, D. C. Rose, M. I. Rose, Sister Mary Charlotte, R. H. Sprague, Guy Stevenson, Louise C. Stolle, J. T. Vallandingham, J. A. Ward, R. H. Wilson, Jr., T. M. Wright, W. M. Zaring.

The following officers were elected for the year 1954–55: Chairman, Professor Charles Hatfield, Georgetown College; Secretary-Treasurer, Professor A. W. Goodman, University of Kentucky; Traveling Lecturer, Professor Aughtum S. Howard, Kentucky Wesleyan College.

The following papers were presented:

1. *Number systems*, by Professor J. A. Ward, University of Kentucky.

Number systems to various bases are discussed. The study of such systems makes clear the properties and methods of arithmetic in our ordinary decimal system. Special attention is given to the binary system and its use in electronic digital computers.

2. *Upper limits to real roots of polynomial equations*, by Mr. Eugene Bradley, Georgetown College, introduced by Professor Charles Hatfield.

In locating the real roots of a polynomial equation, it is frequently valuable to use theory of upper limits to aid in the solution. This is a discussion of several theorems on upper limits and a presentation of examples to illustrate their application.

3. *A problem in weighing*, by Professor Guy Stevenson, University of Louisville.

In general find, for a given positive integer $N$, the minimum number of positive integers $w_1, w_2, \cdots, w_n$ such that $w_1+\cdots+w_n=N$ and such that each integer from 1 to $N$ can be expressed as an algebraic sum of some subset of the $w$'s. It was shown that if $(3^{n-1}-1)/2<N \leq(3^n-1)/2$, the minimum number of $w$'s is $n$, and if $N=(3^n-1)/2$, there is only one set of $w$'s and they are $1, 3, 3^2, \cdots, 3^{n-1}$. A method was discussed for finding how many sets of $n$ integers may be determined for a given number $N$. In general, quite a few sets are possible.

4. *Women in mathematics*, by Miss Mary A. Eschrich, Nazareth College, introduced by Sister Mary Charlotte.

Among the many minor mathematicians, there are six women who have been recognized by historians and tradition as most interesting contributors to the mathematical heritage. The lives of fourth century philosopher and astronomer Hypatia, Mary Gaetana Agnesi, eighteenth century Italian student of calculus, Sophie Germain, the scientist who experimented with mathematical physics, England's distaff mathematician, Mary Somerville, and the friend and pupil of Herman

Weierstrass, Sonya Kovalevski, clearly demonstrate the drama and novelty found in the long history of mathematics and mathematicians.

5. *Equation skeletons*, by Professor J. C. Eaves, University of Kentucky.

The equation $AX=K$, where $X$ is the column vector having elements $X_1, X_2, \cdots, X_n$ and $A$ is an $m$ by $n$ matrix, is solved for the $x_i$ by using only the augmented matrix as the skeleton of the system and performing row transformations to obtain equivalent systems and eventually the equivalent system $CX=H$, where $C$ is the appropriate rank canonical form or supplementary rank canonical form. From this the solution is immediately read or the conditions under which a solution exists are apparent.

6. *The mathematics a student should know to enter college*, by Professor Beulah Graham, Campbellsville College.

The widespread lack of background for college freshman mathematics and the accompanying poor study habits due to the lack of required homework in high school are cited. There is obviously no improvement in the readiness of freshmen for college mathematics in the past ten years. However, the colleges are trying to help the situation by introducing a course in basic mathematics and by giving help to the most promising students.

7. *A useful transformation in the theory of binary star orbits*, by Professor R. H. Wilson, Jr., University of Louisville.

The classical orbital elements for binary stars are polar coordinate parameters; equations using them are trigonometric and ill-adapted to machine computation. New "Theele-Innes" analytical geometrical elements $A$, $B$, $F$, and $G$ are direction numbers of the major and minor axes of the true orbit referred to the south-north and west-east lines. Necessary tables of elliptical rectangular coordinates $(X, Y)$ referred to the major axis and latus rectum of the true orbit have been published. Resulting equations for the apparent rectangular coordinates $a=AX+FY$ and $y=BX+GY$ are simply a transformation of coordinates and are convenient for use with calculating machines.

8. *Analytic extension of functions defined by Dirichlet series*, by Mr. D. C. Rose, University of Kentucky.

The author gave some remarks contrasting the analysis of Taylor series with that of Dirichlet series. He then sketched the method employed to obtain very general results on the analytic extension of functions defined by general Dirichlet series.

9. *The conference at Boulder*, by Professor D. W. Pugsley, Berea College.

General, interesting, and personal remarks concerning the author's stay at the conference made up this paper.

10. *Summations*, by Mr. J. B. Cornelison, University of Kentucky.

Several theorems in sums of products of binomial coefficients such as

$$\sum_{k=p}^{n} (-1)^k \binom{k}{p}\binom{n}{k} = 0 \qquad \text{for } p < n,$$

were given. A different method of proof was given for each theorem in order to illustrate a few of the many different methods available.

11. *Some questions of current interest concerning the teaching of mathematics*, by Professor E. A. Cameron, University of North Carolina. (By invitation.)

This invited lecturer discussed each of the elementary college mathematics courses, raising

questions as to content and method for each course. Throughout the paper, new subject matter and new aims and purposes for the courses were stressed.

W. J. ROBINSON, *Secretary*

## THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Maryland, College Park, Maryland, on May 1, 1954. Professor Daniel C. Lewis, Jr., Chairman of the Section, presided at the morning and afternoon sessions.

There were ninety-six persons in attendance, including the following sixty-eight members of the Association.

J. C. Abbott, D. F. Atkins, R. P. Bailey, David Blackwell, R. O. Blummer, Jr., J. W. Brace, B. H. Buikstra, J. M. Cameron, H. H. Campaigne, B. R. Cato, Jr., H. J. Cheston, Jr., G. R. Clements, E. W. Coffin, Elizabeth H. Cuthill, Melchior Di Carlo-Cottone, R. P. Eddy, Gloria C. Ford, C. H. Frick, S. I. Gass, B. C. Getchell, Mildred W. Going, Michael Goldberg, R. A. Good, E. S. Grable, E. C. Gras, Donald Greenspan, D. W. Hall, A. J. Hoffman, E. C. Hubbard, M. A. Hyman, S. B. Jackson, F. E. Johnston, R. H. Kasriel, L. M. Kells, F. B. Key, Rev. C. F. Koehler, Karl Kozarsky, D. C. Lewis, Jr., D. B. Lloyd, M. M. Lotkin, Elizabeth C. Lukacs, Carol V. McCamman, M. H. Martin, Joseph Milkman, L. I. Mishoe, R. W. Moller, T. W. Moore, Morris Newman, W. H. Norris, P. L. Oglesby, M. W. Oliphant, Walter Penney, G. W. Petrie, III, W. W. Proctor, O. J. Ramler, R. W. Rector, Conrad Rennemann, Jr., J. N. Rice, W. G. Rouleau, J. W. Sawyer, C. H. Sisam, W. A. Soaper, Jr., W. J. Strange, Choy-Tak Taam, J. A. Tierney, P. M. Whitman, J. W. Wrench, Jr., D. M. Young, Jr.

The following officers were elected to serve for a period of one year: Chairman, Professor C. H. Frick, Mary Washington College of the University of Virginia; Vice-Chairmen, Miss Carol V. McCamman, Calvin Coolidge High School, Washington, D. C., and Professor M. W. Oliphant, Georgetown University; Secretary, Professor R. P. Bailey, United States Naval Academy. The section voted that a registration fee of twenty-five cents be charged at future meetings to cover costs of future activities of the section. Professor W. H. Norris reported that approximately 1700 students from 46 high schools were scheduled to take part in the section's first high school contest the following week. The secretary announced that at least six lectures had been given in the lecture series for undergraduates sponsored by the section.

The following papers were presented:

1. *Some practical and theoretical applications of the transportation problem*, by Dr. A. J. Hoffman, National Bureau of Standards, Washington, D. C.

The linear programming problem of minimizing $\sum_{i,j} c_{ij}x_{ij}$, $C=(c_{ij})$ being a given $m$ by $n$ matrix and $X=(x_{ij})$ being a variable $m$ by $n$ matrix with non-negative entries and prescribed row and column sums, is known as the transportation problem. Practical applications include: (1) the cheapest shipment of a commodity from various sources to required destinations, (2) most effective assignment of personnel to tasks, (3) evaluation of bids in the awarding of contracts. Theoretical applications include: (1′) various theorems on eigenvalues and singular values of matrices, (2′) alternative proofs of theorems about distinct representatives of subsets. Some of the problems being solved for government agencies were discussed.

2. *Random numbers for high speed computers*, by Mr. J. M. Cameron and Dr. Morris Newman, National Bureau of Standards, Washington, D. C., presented by Dr. Newman.

The use of random numbers in Monte Carlo methods and the solution of statistical problems by sampling experiments has led to a search for simple methods of generating sequences of random numbers on high speed computers. To be useful, such methods must possess simplicity and speed while satisfying certain statistical criteria. Tests for various methods have been programmed by the Bureau of Standards for their electronic digital computer in Washington. These tests provide checks on frequency distribution, moments, serial correlations, and randomness properties such as number of ascending and descending runs. One method tested, involving additions only, is particularly fast.

3. *On the limit of the coefficients of a certain eigenfunction series*, by Professor Luna Mishoe, Morgan State College.

The problem of expanding a function $f(x)$ of bounded variations over the interval (0, 1) in terms of $u_n$, the eigenfunctions of $u''+q(x)u+\lambda(p(x)u-u')=0$, $u(0)=u(1)=0$, was studied. The resulting series together with conditions for convergence were given. It was shown that the coefficients of $u_n$ in the series approach zero as $n$ approaches infinity if and only if $f(0)=f(1)=0$ for $p$ and $q$ constant.

4. *The propagation of error in numerical integrations*, by Dr. M. M. Lotkin, Ballistic Research Laboratories, Aberdeen Proving Ground.

From the theoretical as well as practical point of view it is of importance to be aware of the stability properties of numerical methods of integration, and growth of error in the large. Proceeding along lines similar to those employed by H. Radamacher, H. Rutishauser, L. H. Thomas, and others, criteria have been developed for stability of a number of well known methods of integrating ordinary differential equations. Using proper values of the pertinent characteristic equations, expressions were derived for the propagation of error due to truncation and rounding.

5. *Arcs of monotone curvature with prescribed end conditions*, by Professor S. B. Jackson, University of Maryland.

Let $A$ and $B$ be two points of the plane which lie respectively on two directed circles $\mathcal{A}$ and $\mathcal{B}$, a line being considered a special case of a circle. The problem is that of drawing an arc $AB$ whose curvature is monotone and which has the given directed circles $\mathcal{A}$ and $\mathcal{B}$ as osculating circles at the endpoints. Special attention was devoted to the question of the existence of such an arc with, in some sense, a minimum of angular measure. Necessary and sufficient conditions were obtained for the existence of such an arc.

6. *Controlled random walks*, by Professor David Blackwell, Howard University.

This was the invited lecture. In a sequence of plays of a two person finite game with a payoff matrix whose elements are vectors in $k$-space, the extent to which either player can control the average payoff vector in a long series of plays was investigated. In particular every closed convex set $S$ in $k$-space has one of two properties: either (1) player I can force the average payoff to approach $S$ with probability 1 as the number of plays becomes infinite, or (2) there is a closed convex set $T$ disjoint from $S$ such that player II can force the average payoff to approach $T$ with probability 1 as the number of plays becomes infinite.

C. H. FRICK, *Secretary*

## THE MAY MEETING OF THE MISSOURI SECTION

The annual spring meeting of the Missouri Section of the Mathematical Association of America was held jointly with the Missouri Council of Teachers of Mathematics at the University of Missouri, Columbia, Missouri, on May 7, 1954. Professor W. R. Utz, Vice-Chairman of the Section, and Professor Margaret F. Willerding, Associate Secretary of the Section and Chairman of the Missouri Council of Teachers of Mathematics, presided in turn at the morning session. Professor G. H. Jamison, Chairman of the Section, presided at the afternoon session.

There were eighty persons in attendance, including the following thirty-eight members of the Association:

L. W. Akers, J. J. Andrews, S. Louise Beasley, L. M. Blumenthal, P. B. Burcham, W. A. Couch, Mary L. Cummings, C. H. Dalton, J. F. Daly, H. Margaret Elliott, D. H. Erkiletian, Jr., G. M. Ewing, C. V. Fronabarger, Nola L. A. Haynes, F. F. Helton, N. Q. Hubbard, C. A. Hutchinson, G. H. Jamison, C. A. Johnson, L. O. Jones, C. E. Kelley, R. E. Lee, Adele Leonhardy, F. H. Lloyd, Jessie H. McLean, R. J. Michel, R. R. Middlemiss, Marie A. Moore, L. E. Pummill, R. M. Rankin, Francis Regan, Lois J. Roper, Sister M. Pachomia, J. H. Skelton, Sarah D. Springer, W. A. Vezeau, Margaret F. Willerding, J. L. Zemmer.

At the business meeting the following officers were elected for the coming year: Chairman, Professor Maria Castellani, University of Kansas City; Vice-Chairman, Professor C. H. Dalton, Southeast Missouri State College; Secretary-Treasurer, Sister M. Pachomia, College of St. Teresa. Professor Margaret F. Willerding, Harris Teachers College, was elected to serve as Associate Secretary for a period of five years.

It was voted that the Sectional Governor become a member of the Executive Committee. It was decided that the committee appointed last year to study the requirements of the State Department of Education with respect to mathematics continue to serve and follow up the work done during the past year.

The following recommendations were made by the above committee:

"We recommend the following requirements for certifying teachers of mathematics in high school:

1. At least fifteen semester hours of college mathematics, including five hours of college algebra and trigonometry. The college algebra course shall have a prerequisite of one and one-half units of high school algebra or the equivalent. Five hours of analytic geometry or analytic geometry and calculus may be substituted for the college algebra and trigonometry;

2. Teachers of plane geometry and/or solid geometry shall have at least two and one-half hours of college geometry, modern geometry or advanced geometry;

3. Teachers of trigonometry shall have at least five hours of analytic geometry or analytic geometry and calculus."

The following program was presented:

1. *Some remarks concerning tetrahedra*, by Professor L. M. Blumenthal, University of Missouri.

A sextuple of positive numbers are the lengths of the edges of at most thirty tetrahedra, no two of which are metrically congruent. Procedures for obtaining sextuples for which the maximum is realized (maximal tetrahedral sextuples) were presented. Three problems arise concerning tetrahedral-producing operations on a given pair $T_3(p_1, p_2, p_3, p_4)$, $T_3(q_1, q_2, q_3, q_4)$ of tetrahedra.

(1) Determine functions $f$ (called of Type I) such that four points $r_1, r_2, r_3, r_4$ of euclidean three-space exist with $r_ir_j = f(p_ip_j, q_iq_j)$, $(i, j = 1, 2, 3, 4)$.

(2) Determine functions $f$ of Type II such that the numbers $f(p_ip_j, q_iq_j)$, $(i, j = 1, 2, 3, 4)$ are tetrahedral.

(3) Determine functions $f$ of Type III such that $f(p_ip_j, \gamma(q_iq_j))$ $(i, j = 1, 2, 3, 4)$ are tetrahedral, where $\gamma$ is a given correlation between the two sextuples $p_ip_j$, $q_iq_j$, $(i, j = 1, 2, 3, 4)$.

It was shown that $(x^2+y^2)^{1/2}$ and $(x+y)^{1/2}$ are functions of Type I, while if $T_3(q_1, q_2, q_3, q_4)$ is regular, then $f(x) = x+k$ is of Type I, where $q_iq_j = k > 0 (i \neq j)$. The function $x+u$ is not of Type I for arbitrary tetrahedras. These results are all valid for $n$-simplices.

2. *Mathematics in general education*, by Professor J. A. Seeney, Lincoln University, introduced by the secretary.

There are many concepts regarding the nature of General Education. Underlying all these ideas is the notion that we must educate the whole man. Many conflicts are evident today as to what this education should be.

General Education is defined as that education needed for the cohesion of a democratic society. It is concerned with the social needs of the layman. High schools and colleges must carry out this function in addition to others imposed by the nature of the society.

We can best meet the mathematics needed by the layman in a required course in general mathematics. The teachers and experts must determine what mathematics is needed in everyday life and teach it so that it has meaning to the student and stresses the important mathematical relationships.

General mathematics should be required of all students. We have no assurance that mathematical skills will function in everyday life unless we teach it that way.

3. *The Missouri traveling exhibit* by Miss Frances Story, St. Charles High School, introduced by the secretary.

The Missouri Council of Teachers of Mathematics is unique in having prepared a traveling exhibit of teaching aids and ideas that would be helpful to the teaching of mathematics. These aids consist of teaching aids for all levels of mathematics instruction—from charts for teaching the number concept in the primary grades to devices for use in plane geometry, trigonometry and advanced algebra in the secondary school. These aids are either student or teacher made and can be copied with little or no expense by using materials at hand.

A teacher has access to the traveling exhibit by requesting it and paying the mailing expense.

The teachers who have compiled the exhibit do not advocate that teaching aids and ideas, such as those in the exhibit, take the place of the normal classroom methods, but that they be used in addition to the regular teaching procedure.

4. *General mathematics—pupil attitude and teacher understanding*, by Mr. E. J. Jackson, St. Louis Public Schools, introduced by the secretary.

The data from a survey of a large number of ninth grade pupils in St. Louis high schools indicate that General Mathematics is looked upon as a valuable vocational tool as well as an important part of one's general education. These data have implications for teachers of the subject with respect to their techniques and philosophy of the subject.

A study of teacher understanding of arithmetic in the Metropolitan Area of St. Louis reveals a serious lack of understanding of the processes, concepts and short-cuts in the mathematics they teach. The evidence is strong that teachers have learned their arithmetic by rote and teach arithmetic concepts as they learned them.

5. *Mathematics in Missouri schools*, by Mr. I. F. Coyle, State Department of Education, introduced by the secretary.

Curricular requirements in mathematics for pupils in elementary and secondary schools are discussed, as are the requirements for the certification of teachers of mathematics in the public schools.

Comparison is noted between Missouri secondary school mathematics enrollments in 1952–53 and those of 1948–49. The comparison shows a slight falling in mathematics on an elective basis.

Possibilities for improving instruction in mathematics in elementary and high schools are described. The purposes and policies of teacher certification are enumerated, and the procedures for determining certification requirements in Missouri are discussed.

6. *Report of the committee appointed to study improvement of mathematical education* by Professor R. J. Michel, Southeast Missouri State College.

7. *A serious look at college mathematics*, by Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado. (By invitation.)

The speaker discussed informally the present day difficulties of college mathematics programs, with respect to the preparation of students, the courses, the text books, using observations of experience at the University of Colorado as basis.

MARGARET F. WILLERDING, *Associate Secretary*

## THE MAY MEETING OF THE WISCONSIN SECTION

The twenty-second annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Wisconsin State College, Eau Claire, Wisconsin, on May 8, 1954. Professor L. P. Wahlstrom, Chairman of the Section, presided.

There were fifty-three present including the following twenty-five members:

R. H. Bing, Leonard Bristow, G. L. Bullis, W. L. Duren, Jr., J. V. Finch, C. E. Flanagan, Harold Glander, W. A. Golomski, E. G. Harrell, R. C. Huffer, E. R. Johnston, A. P. Loomer, C. C. MacDuffee, J. R. Mayor, J. J. McLaughlin, Elli Otteson, C. L. Rich, Sister Mary Corona, Sister M. Domitilla, Sister Mary Felice, Sister Mary Petronia, E. W. Swokowski, C. J. Vanderlin, Jr., R. D. Wagner, and L. F. Wahlstrom.

The following officers were elected for the coming year: Chairman, Dr. A. E. May, Racine Extension of the University of Wisconsin; Vice-Chairman, Mr. C. E. Flanagan, Wisconsin State College, Whitewater; Secretary-Treasurer, Sister Mary Felice, Mount Mary College; Program Committee: Professor R. C. Buck, University of Wisconsin, Professor J. C. Stewart, Lawrence College.

The principal item on the agenda of the business meeting was action on the revision of the by-laws of the section. A committee under the chairmanship of Professor H. P. Pettit presented a proposal for such a revision and the report, read by the secretary in the absence of Professor Pettit, was adopted with a slight amendment.

The program was as follows:

1. *Orientation*, by Dr. Leonard Haas, Dean of Instruction, Wisconsin State College, Eau Claire.

2. Panel discussion: *What is going on in Wisconsin colleges?* by Professor J. R. Mayor, University of Wisconsin, Professor R. C. Huffer, Beloit College, and Dr. J. J. McLaughlin, Wisconsin State College, River Falls.

Professor Mayor stated that the major in mathematics in the School of Education, The University of Wisconsin, is required to complete a year of calculus and four mathematics courses (of three semester hours each) at the junior and senior level. A year's course in physics is also required. The student must maintain a grade-point average of 1.3 (A, 3 points; B, 2 points; and C 1 point) in mathematics courses counted on the major. The methods course is given as an education course. The supervision of student teaching and the methods course are in charge of a staff member with academic rank in both the Department of Mathematics and the Department of Education.

On the basis of questionnaires sent to seventeen private colleges and universities in Wisconsin, Professor R. C. Huffer described a "modal" private college. Deviations from the mode and the variety of course offerings were then discussed. Particular attention was given to introductory and integrated courses in the freshman year, and to courses in statistics and the mathematics of finance.

Questionnaires, similar to those sent by Dr. Huffer, were sent to the State colleges, and the results of these were summarized by Dr. J. J. McLaughlin.

3. Panel discussion: *Articulating the high school and college programs in mathematics*, moderated by Miss Dorothy Sward, Roosevelt Junior High School, Racine; Leaders: Mr. Sidney Ainsworth, Wisconsin High School, Madison; Mr. Laurence Cook, Eau Claire High School; Miss Laura Wagner, Fort Atkinson, Wisconsin; Miss Elli Otteson, Eau Claire High School.

4. *Boundary conditions for any new college mathematics*, by Professor W. L. Duren, Jr., Tulane University. (By invitation.)

The most important of the boundary conditions, which apply to the problem of finding an appropriate form of college mathematics for freshmen, is that the subject must be interesting to young mathematicians, who we hope will teach it, as well as to students. In order to survive in the curriculum, our subject must become a part of the student's working mental equipment. These conditions affecting the cultural approaches to mathematics are compatible with the highest cultural aims.

Educational conditions applying to mathematics are: school-college articulation, accommodation to the abilities of average students, advancement of the more able ones, the training of teachers, and the criterion that it is unwise to force immature students upon entering college to make choices which later will restrict their freedom of development. Science and engineering impose conditions of support for the existing technology and for the attack on unsolved problems of science, particularly in social science, which has been relatively neglected by mathematicians.

All this leads us to try to formulate one universal freshman course which could be supplemented by technical "laboratory" work for particular scientific requirements. The wisdom, insight, writing and teaching of the best mathematicians are needed to guide this movement along lines which are true to the development of mathematics.

5. *Summary of the discussions*, by Professor R. H. Bing, University of Wisconsin.

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## CALENDAR OF FUTURE MEETINGS

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pennsylvania, May, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May, 1955.

IOWA, St. Ambrose College, Davenport, April 15–16, 1955.

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 18–19, 1955.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D.C. December 4, 1954.

METROPOLITAN NEW YORK, Queens College, Flushing, New York, April 30, 1955.

MICHIGAN, Michigan State College, East Lansing, March 26, 1955.

MINNESOTA, University of Manitoba, Winnipeg, October 16, 1954.

MISSOURI, University of Kansas City, Spring, 1955.

NEBRASKA

NORTHERN CALIFORNIA, University of California, Berkeley, January 15, 1955.

OHIO

OKLAHOMA, Oklahoma City University, October 29, 1954.

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 17, 1955.

PHILADELPHIA, Princeton University, Princeton, New Jersey, November 27, 1954.

ROCKY MOUNTAIN, University of Wyoming, Laramie, Spring, 1955.

SOUTHEASTERN, Tennessee Polytechnic Institute, Cookeville, March 11–12, 1955.

SOUTHERN CALIFORNIA, Santa Monica City College, March 12, 1955.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1955.

TEXAS, Abilene Christian College, Abilene, April, 1955.

UPPER NEW YORK STATE, University of Buffalo, May 14, 1955.

WISCONSIN, Cardinal Stritch College, Milwaukee, May, 1955.

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